MORU Cardington Technical Note No. 14

An error analysis of the Cardington turbulence probe orientation algorithm

by

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4 December 1992

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RAF Cardington
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Beds, MK42 0TH

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1 Introduction

The current generation of turbulence probes used by the MORU at Cardington differ from earlier designs by including inclinometers to measure the probe orientation, thus allowing winds measured relative to the probe body to be transformed into a fixed geographically based co-ordinate system. In earlier designs of turbulence probes, used from the late sixties, the wind sensors were mounted on a damped pendulum to keep them oriented to the local vertical. However, accelerations of the probe, caused by movements of the balloon, mean't that instantaneously the pendulum could be at an angle to the vertical, leading to errors in the measurement of wind fluctuations, particularly in strong wind conditions. The acceleration of the probe still limits the frequency band for which the inclinometers can be used to provide useful information on probe orientation but, by taking advantage of the relatively slow motion of the balloon, it is possible to use the inclinometer data, combined with data from a three axis magnetometer to give the instantaneous probe orientation.

The quality of the turbulence data obtained using the turbulence probe system depends crucially on the accuracy with which the orientation of the probe can be calculated and on the sensitivity of the algorithm to errors in the measurements. This note presents the results of a linearised error analysis of the algorithm presently used and estimates the effects that typical measurement errors may have on turbulence statistics.

2 Analysis of probe orientation

A turbulence probe consists of a rectangular package containing electronics, with gill anemometers and various thermometers attached to the outside and a wind vane which keeps the package pointing into wind. The probe is clamped to the steel tether cable of a helium filled balloon and is free to rotate about the cable. Pitch and roll inclinometers are mounted in the package while a three-axis flux gate magnetometer is mounted in a small box attached to the vane. The magnetometer is mounted outside the package to remove it from small magnetic fields associated with the electronics. Lapworth and Mason (1989) discuss the design and performance of the probe in detail.

An arbitrary probe orientation, relative to some reference orientation, can be described using three angles which give rotations about three different axes. The reference orientation is chosen so that the probe z-axis is along the local vertical, the probe x-axis points in the direction of the Earth's magnetic field (i.e. magnetic North) and the y-axis is orthogonal to these two axes and chosen so that the three axes form a right handed co-ordinate system. Figure 1 shows the definitions of the coordinate axes. With the data that are available from the inclinometers and the magnetometers an arbitrary orientation of the probe can be specified by three rotations;

A1. A rotation about the probe z-axis through an angle θ.

A2. A rotation about the probe y-axis through an angle ϕ.

A3. A rotation about the probe x-axis through and angle ψ.

The directions of the angles are taken to conform to the usual right handed convention and are indicated in Figure 1.

The pitch and roll angles, ϕ and ψ, are measured directly by the inclinometers. The azimuth angle, θ, can be obtained from;

$$[P]^{-1} \times [R]^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = [A] \begin{pmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{pmatrix} \tag{1}$$

where:

[A], [P] and [R] are the matrices describing the three rotations through the angles θ, ϕ and ψ as defined above. The components of these matrices are given in Appendix 1.

α, β and γ are the x, y and z outputs of the magnetometer, M_x, M_y and M_z, normalised so that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

δ is the dip angle of the Earth's magnetic field. From the definition of the reference co-ordinate system $B_y = 0$.

If the turbulence probe were not subject to accelerations the angles θ, ϕ and ψ would be sufficient to specify the instantaneous probe orientation. However, for frequencies $\geq 1Hz$ the inclinometer outputs are affected by accelerations of the probe, which cause errors in the instantaneous orientation if no filtering is applied to the inclinometer data. Simple low pass filtering of the inclinometer data is not sufficient because real changes of the probe orientation do take place on a timescale of 1 second. The solution to this problem is found from the observation that while the tangent vector to the cable at the probe changes with time it does so relatively slowly. Since the probe z-axis is parallel to the cable this suggests that a different set of angles might be better for describing the probe oriention. Starting with the probe in the reference orientation the angles are defined as follows, B1. Rotate through an angle n about the probe z-axis.

B2. Rotate through an angle m about the probe y-axis.

B3. Rotate through an angle l about the probe z-axis.

The three angles l, m and n are the Euler angles which are frequently used to describe the motion of rigid bodies in mechanics (Kibble).

The components of the probe z-axis relative to the reference co-ordinate system are,

$$\begin{pmatrix} \sin m \cos n \\ \sin m \sin n \\ \cos m \end{pmatrix} = [A]^{-1} \times [P]^{-1} \times [R]^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

multiplying out the RHS of equation 3 gives

$$\begin{pmatrix} \sin m \cos n \\ \sin m \sin n \\ \cos m \end{pmatrix} = \begin{pmatrix} \cos\theta \sin\phi \cos\psi + \sin\theta \sin\psi \\ \sin\theta \sin\phi \cos\psi - \cos\theta \sin\psi \\ \cos\theta \cos\psi \end{pmatrix} \tag{3}$$

Equating components and after some straightforward algebra equation 2b gives:

$$\cos m = \cos\phi \cos\psi \tag{4}$$

$$\tan n = \frac{(\sin\phi \tan\theta - \tan\psi)}{(\sin\phi + \tan\psi \tan\theta)} \tag{5}$$

The angle l is obtained from;

$$[L]^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = [M] \times [N] \begin{pmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{pmatrix} \tag{6}$$

[L], [M] and [N] are matrices describing the three rotations defined in B1-B3 above. The components of these matrices are given in appendix 1.

Multiplying out and equating components gives;

$$
\begin{aligned}
\cos l &= \frac{(\alpha(\cos m \cos n \cos \delta + \sin m \sin \delta) - \beta \sin n \cos \delta)}{(\alpha^2 + \beta^2)} \\
\sin l &= -\frac{(\beta(\cos m \cos n \cos \delta + \sin m \sin \delta) + \alpha \sin n \cos \delta)}{(\alpha^2 + \beta^2)}
\end{aligned} \tag{7}
$$

The angles m and n give the orientation of the cable at the probe and vary slowly with time. They can be low pass filtered to remove those frequencies that are contaminated by the probe acceleration. The angle l, which gives the rotation of the probe about the cable, depends on these the angles m and n and the data from the magnetometer, which are unaffected by the probe acceleration.

Given the angles l, m and n all that is required to begin the error analysis is the relationship between the components of a vector relative to the probe co-ordinate frame and the components of the same vector with respect to the reference co-ordinate frame. This can be obtained from;

$$
\begin{pmatrix} V_x^f \\ V_y^f \\ V_z^f \end{pmatrix} = [N]^{-1} \times [M]^{-1} \times [L]^{-1} \begin{pmatrix} V_x^p \\ V_y^p \\ V_z^p \end{pmatrix} \tag{8}
$$

Where the superscrips f and p refer respectively to the reference and probe co-ordinate frames.

3 Error analysis

For this analysis it will be assumed that the measured pitch and roll angles are subject to small errors $\Delta\phi$ and $\Delta\psi$ and that the three field components measured by the flux gate magnetometers have errors ΔM_x, ΔM_y and ΔM_z. The errors can be supposed to result from small angular misalignments between the various components of the probe. These measurement errors will result in errors in the probe orientation angles of Δl, Δm and Δn which will produce errors in the wind data obtained by the probe. The purpose of this paper is to establish how large the errors in wind measurements are and how they vary with wind direction. The errors will be assumed to be small so that a linear analysis can be carried out. For small variations in l, m and n equation 8 gives;

$$\begin{pmatrix} \Delta V_x^f \\ \Delta V_y^f \\ \Delta V_z^f \end{pmatrix} = (\Delta[N]^{-1}\times[M]^{-1}\times[L]^{-1}+[N]^{-1}\times\Delta[M]^{-1}\times[L]^{-1}+[N]^{-1}\times[M]^{-1}\times\Delta[L]^{-1}) \begin{pmatrix} V_x^p \\ V_y^p \\ V_z^p \end{pmatrix}$$

$$+ [N]^{-1} \times [M]^{-1} \times [L]^{-1} \begin{pmatrix} \Delta V_x^p \\ \Delta V_y^p \\ \Delta V_z^p \end{pmatrix} \tag{9}$$

$$= [E] \begin{pmatrix} V_x^p \\ V_y^p \\ V_z^p \end{pmatrix} + [R] \begin{pmatrix} \Delta V_x^p \\ \Delta V_y^p \\ \Delta V_z^p \end{pmatrix} \tag{10}$$

Where the definition of the matrices E and R can be obtained by comparing the equations 9 and 10. The second term in equations 9 and 10 arises from any errors there may be in the measurment of the vector relative to the probe, e.g errors in the Gill anemometers for the wind vector. The error matrices, $\Delta[L]^1$ etc. can be written;

$$\Delta[L]^{-1} = [L_\epsilon^{-1}]\,\Delta l$$

where the components of the matrix $[L_\epsilon{}^{-1}]$ are,

$$[L_\epsilon^{-1}]_{ij} = \frac{\partial[L]_{ij}^{-1}}{\partial l} \tag{11}$$

with similar expressions for $\Delta[M]_{ij}^{-1}$ and $\Delta[N]_{ij}^{-1}$

Equations 4, 5, 6 and 7 can be used to relate changes in l, m and n to small changes in θ, ϕ and ψ. After some algebra the results are.

$$\sin m\Delta m = \sin\phi\cos\psi\Delta\phi + \cos\phi\sin\psi\Delta\psi \tag{12}$$

$$\Delta n = \Delta\theta + f_\phi(\theta, n)\Delta\phi - f_\psi(\theta, n, \phi, \psi)\Delta\psi \tag{13}$$

$$\Delta l = (g_\alpha(m,n,l,\delta)\Delta\alpha + g_\beta(m,n,l,\delta)\Delta\beta - g_m(m,n,l,\delta,\alpha,\beta)\Delta m$$
$$-g_n(m,n,l,\delta\alpha\beta)\Delta n - g_\delta(m,n,l,\delta,\alpha,\beta)\Delta\delta)/(\alpha^2+\beta^2) \quad (14)$$

The various functions f_i, g_i in these expressions are given in the appendix. The error in the azimuth angle can be obtained from equation 1 as;

$$\Delta[A]\begin{pmatrix}\cos\delta\\0\\-\sin\delta\end{pmatrix} - [A]\begin{pmatrix}\sin\delta\\0\\\cos\delta\end{pmatrix}\Delta\delta = [P]^{-1}\times\Delta[R]^{-1}\begin{pmatrix}\Delta\alpha\\\Delta\beta\\\Delta\gamma\end{pmatrix} +$$
$$(\Delta[P]^{-1}\times[R]^{-1} + [P]^{-1}\times\Delta[R]^{-1})\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \quad (15)$$

Equation 15 also provides the error in the measured dip angle, $\Delta\delta$, to be used in equation 14. It is implicit in the derivation of these errors that the calculated dip angle is used, since if a prescribe dip angle is used in the expression for the azimuth angle the orientation problem is over constrained: the number of independent measurements being two (any two of α, β or γ), with only one quantity derived, i.e. the probe azimuth.

A point to note in equation 14 is that $\Delta 1 \propto 1/(\alpha^2+\beta^2)$. If the z-axis is nearly parallel to the magnetic field vector the magnitude of the component in the probe x-y plane, $(\alpha^2+\beta^2)^{\frac{1}{2}}$ becomes small and the error in l can become very large. In fact the angle l cannot be calculated at all if $(\alpha^2+\beta^2)^{\frac{1}{2}} = 0$, which occurs when $m = 90° - \delta$ and $n = 180°$ i.e. when the cable is parallel to the magnetic field lines. If the probe pitch is less than 23° there are two wind directions, corresponding to two values of the roll angle, for which $(\alpha^2+\beta^2) = 0$ The same problem does not occur with the angles m and n which means that even if $(\alpha^2+\beta^2)^{\frac{1}{2}}$ is small the data from the probe can still be used in the calculation of the motions of probes at greater altitudes.

Equations 9, 10, 12, 13 and 14 can be combined to calculate the error in components of a vector, measured relative to the reference frame, which result from errors in the inclinometer and magnetometer data. The vector of most interest is the wind vector and the main purpose of this note is to calculate the error in turbulence statistics for specified errors in the inclinometers and magnetometers. In order to proceed a relationship between the wind vector and probe orientation is required. These are not independent quantities since the probe is free to rotate about the balloon tether cable and is kept pointing into wind by the vane. The angles m and n describing the direction of the cable tangent vector are determined by the wind at the level of the balloon and can reasonably be taken to be independent of the wind at the probe, if the two are well separated.

Assuming that the x-axis of the probe frame is always parallel to the wind vector, i.e the y-component of the wind in the probe frame is zero, and that the x-component of the wind in the probe frame is negative, equation 8 can be used to derive a relationship between the wind direction and the angle l, given the angles m and n. The assumption that the probe x-axis is always parallel to the wind vector restricts the validity of the present results to situations where the proportion of the turbulence energy and stress associated with wavelengths shorter than the length constant of the wind vane (~10 m) are small. Splitting the terms in equation 8 into mean and fluctuating parts, and recalling the assumption that m and n are constant then, to first order in the fluctuations;

$$V^f + v^f = [N]^{-1} \times [M]^{-1} \times ([L]^{-1}V^p) + [L]^{-1}v^p + [l]^{-1}V^p) \qquad (16)$$

The upper case quantites represent means and the lower case ones fluctuations. The vectors are represented by single symbols rather than in full component form for convenience.

Averaging equation 16,

$$V^f = [N]^{-1} \times [M]^{-1} \times [L]^{-1} V^p \qquad (17)$$

The error equation for equation 17 can be obtained from equation 9 and 10 by setting the second term to zero, i.e. assuming that the error in the measurement of the wind vector relative to the turbulence probe by the Gill anemometers is negligible. Subtracting equation 17 from equation 16 gives an equation for the fluctuating part of the wind vector, i.e.;

$$v^f = [N]^{-1} \times [M]^{-1} \times ([L]^{-1}v^p + [l]^{-1} V^p) \qquad (18)$$

By combining the terms in the parenthesis brackets equation 18 can be rewritten as;

$$v^f = [N]^{-1} \times [M]^{-1} \times [L]^{-1} \begin{pmatrix} v_x^p \\ V_1^p 1' \\ v_z^p \end{pmatrix} \qquad (19)$$

The vector on the right hand side of equation 19 can be considered to be the wind vector measured relative to the mean probe orientation, rather than the instantaneous orientation. The y-component of this vector emphasises the behaviour of the probe as a wind vane. The error equation for equation 19 is again equations 9 and 10, but now the second term is not zero, since the fluctuations in the angle l are subject to measurement error. Equations 9 and 10 for this case can be rewritten in tensor notation, with summation implied for repeated indices, as;

$$\Delta v_i^f = E_{ik} v_k^p + R_{ik} \delta_{k2} V_1^p \Delta l' \tag{20}$$

Note that by assumption only the y-component wind vector has an associated error, hence the δ_{k2} in the second term. The error in the fluctuating angle can be written;

$$\Delta l' = \Delta l - \overline{\Delta l} = \frac{\partial \Delta l}{\partial l} l' \tag{21}$$

The derivative is at constant m and n. Substituting this into equation 20.

$$\Delta v_i^f = E_{ik} v_k^p + \frac{\partial \Delta l}{\partial l} R_{ik} \delta_{k2} v_2^p \tag{22}$$

Multiplying equation 22 by $v_j^f = R_{js} v_s^p$ and averaging gives;

$$v_j^f \Delta v_i^f = E_{ik} R_{js} v_k^p v_s^p + \frac{\partial \Delta l}{\partial l} R_{ik} R_{js} \delta_{k2} v_2^p v_s^p \tag{23}$$

The error in the ijth component of the Reynolds stress tensor can be obtained by transposing ij in equation 23 and adding the result to equation 23.

Equation 23 describes the errors in the components of the Reynolds stress tensor measured relative to the reference frame. Usually the components of the stress-tensor are measured relative to a frame with the x-axis along the direction of the mean wind. Taking [D] to be a matrix defining a horizontal rotation from the reference frame to the wind aligned frame, the components of the stress-tensor in this latter frame are;

$$[T_{wind}] = [D] \times [T_{ref}] \times [D]^T \tag{24}$$

where [T_{wind}] and [T_{ref}] are the Reynolds stress tensor in the wind aligned and reference frames respectively and $[D]^T$ is the transpose of [D].

For small errors;

$$\Delta[T_{wind}] = \Delta[D] \times [T_{ref}] \times [D]^T + [D] \times [T_{ref}] \times \Delta[D]^T + [D] \times \Delta[T_{ref}] \times [D]^T \tag{25}$$

The first two terms in this equation arise from the error in the wind direction. The last term arises from the errors in the stress tensor relative to the reference frame, which implicitly includes a contribution from the error in the wind direction which should be compensated for by the first two terms.

4 Results

Although the effect of sensor errors on the final turbulence measurements are complex there are some general points that can be made can be made.

1. The variance of the lateral wind component, ${\sigma_v}^2$ appears to be most sensitive to alignment errors. The sensitivity arises from the second term in equation 23.

2. Errors in ${\sigma_u}^2, {\sigma_w}^2$ and $\overline{uw}$ arise mainly from the first term in equation 23. For errors in the pitch inclinometer the errors are what would be expected for a levelling error, with the tilt vector and wind vector in the same plane.

3. For $\overline{vw}$ both terms in equation 23 are important, but they have opposite affects and nearly cancel so that errors in $\overline{vw}$ are generally small.

4. Errors can become very large for probe orientations where $(\alpha^2 + \beta^2)$ is small. In other orientations the errors in ${\sigma_v}^2$ and other quantities are generally less that 10% per degree.

5 Summary

A linear error analysis of the algorithm used to determine the orientation of the Cardington turbulence probe has been presented. As part of the analysis the derivation of the probe orientation from the inclinometer and magnetometer data was also given. This derivation clearly shows that the dip angle calculated from the probe data should be used in calculating the probe orientation if all the orientation data measured by the probe is to be used. The results of the error analysis show;

1. For certain probe orientations the errors in the turbulence statistics become large. These orientations occur when the probe z-axis is nearly parallel to the magnetic field lines.

2. ${\sigma_v}^2$ appears to be affected much more than ${\sigma_u}^2$ by errors in the inclinometer and magnetometer data . This is because ${\sigma_v}^2$ is determined primarily by the rotation of the probe about the tether cable as it is kept pointing into wind by the vane.

3. $\sigma_u{}^2, \sigma_w{}^2$ and $\overline{uw}$ are most affected by errors in the inclinometers, particularly the pitch inclinometer since roll angles are generally small. The errors are essentially tilt errors and may be reduced by performing a co-ordinate rotation to make $\overline{w}=0$.

4. $\sigma_u{}^2, \sigma_w{}^2$ and $\overline{uw}$ are relatively unaffected by errors in the magnetometer data.

The important assumtions made in the analysis were:

1. The probe was assumed to point into the instantaneous wind, i.e. the length constant of the vane was taken to be zero. In practice this will be reasonable if most of the turbulence energy is at wavelengths greater than $\sim 10m$. This may not be the case in stable conditions or near the ground but then the limited response of the Gill anemometers will lead to errors in the turbulence data.

2. The balloon cable was assumed to be stationary. With this assumption the calculated errors depend only on the choice of the Reynolds stress tensor.

3. Errors in the measurement of the wind vector relative to the probe measured by the Gill anemometers were neglected.

4. The effect of flow distortion around the probe was ignored, although this may be an important source of error. Spectra obtained from the turbulence probes do not show the expected 4/3 ratio between the tranverse and longitudinal wind components in the inertial subrange, which may be a due to flow distortion. In principle it is possible to calculate how flow distortion affects turbulence statistics (Wyngard 1981) but because the probe is not symmetric calculating the way in which the potential flow varies with the angle of incidence of the wind is very difficult. More comparisons between the probe and, for example, a sonic anemometer are required to investigate errors due to flow distortion.

6 Appendix

This appendix lists the various matrices used in the analysis and the functions in equations 9b and 9c.

Definition of matrices used in analysis. The inverse matrices are readily obtained and the 'error' matrices can be obtained form equation 11.

$$[A] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [P] = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$[R] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$$

$$[L] = \begin{pmatrix} \cos l & \sin l & 0 \\ -\sin l & \cos l & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [M] = \begin{pmatrix} \cos m & 0 & -\sin m \\ 0 & 1 & 0 \\ \sin m & 0 & \cos m \end{pmatrix}$$

$$[N] = \begin{pmatrix} \cos n & \sin n & 0 \\ -\sin n & \cos n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Functions used in equations 9b and 9c.

$$f_\phi = \sin\psi \cos m / \sin^2 m$$

$$f_\psi = \sin f / \sin^2 m$$

$$g_\alpha = \cos l \sin n \cos\delta + \sin l(\cos m \cos n \cos\delta + \sin m \sin\delta)$$

$$g_\beta = -\sin l \sin n \cos\delta + \cos l(\cos m \cos n \cos\delta + \sin m \sin\delta)$$

$$g_m = (\alpha \sin l + \beta \cos l)(\sin m \cos n \cos\delta - \cos m \sin\delta)$$

$$g_n = \cos m \sin n \cos\delta(\alpha \sin l + \beta \cos l) + \cos n \cos\delta(\beta \sin l - \alpha \cos l)$$

$$g_\delta = (\alpha \sin l + \beta \cos l)(\cos m \cos n \sin\delta - \sin m \cos\delta + \sin n \sin\delta$$
$$(\alpha \cos l - \beta \sin l)$$

7 References

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