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Generalised Lagrangian solutions for atmospheric and oceanic flows

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GENERALISED LAGRANGIAN SOLUTIONS

FOR

ATMOSPHERIC AND OCEANIC FLOWS

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ABSTRACT

Atmospheric or oceanic flows strongly constrained by rotation and stratification can be described by a set of Lagrangian partial differential equations called the semi-geostrophic equations. In these equations the trajectories have to be determined implicitly. Generalised solutions of these equations are defined as a sequence of rearrangements of the fluid, which need not be smooth. These solutions are closely related to generalised solutions of the Monge-Ampere equation. Existence and uniqueness of such solutions is proved. The evolution is shown to be a sequence of minimum energy states of the fluid, giving strong physical plausibilty to the solutions.

Abbreviated title: Generalised Lagrangian solutions.

Key words: semi-geostrophic, Monge-Ampere equation, discontinuities, generalised solutions.

AMS subject classification: 35D05, 76C15.

1. INTRODUCTION

This paper describes a new type of generalised solution of a set of partial differential equations that are important in meteorology and oceanography. In the atmosphere and ocean much of the flow is smooth and large scale, but there are also small very active regions which form an essential part of the large scale dynamics but where the flow is highly complex and often turbulent. The solutions described in this paper simplify the flow in these active regions so that, for instance, a turbulent shear layer becomes a discontinuity. It can then be shown that these simplified solutions can be uniquely determined in terms of the large scale dynamics, in an analogous fashion to the determination of the propagation speed of a hydraulic jump by large scale conservation laws. The simplified solutions are not an accurate approximation to the full equations of motion in the active regions, any more than the one-dimensional shallow water equations are accurate at a hydraulic jump.

The equations used are the semi-geostrophic equations, which are a standard model for slowly varying flows constrained by rotation and stratification. They were introduced by Eliassen [1] and further developed by Hoskins [2]. Salmon [3] has shown how they can be derived from a Hamiltonian formulation. In the simplest case of a coordinate system rotating at a constant rate, the evolution equations can be written in Lagrangian form as a set of ordinary differential equations with no spatial derivatives. The fluid trajectories are determined implicitly, by the requirements of geostrophic and hydrostatic balance. Classical solutions of these equations can be obtained by the geostrophic coordinate transformation introduced by Yudin [4]. It can be shown by using this method that discontinuities can form in a finite time from smooth initial data, Hoskins and Bretherton [5]. They also show that the equations for determining the trajectories can be written as a Monge-Ampere equation.

The analytic solution in geostrophic coordinates derived in [5] can be calculated for all time, but the transformation of it back to physical space becomes multi-valued after the initial formation of a discontinuity, and the results are then unphysical. Cullen and Purser [6] introduced a method of continuing the solutions that made physical sense, and constructed solutions for piecewise constant data by using a geometrical method. Cullen et al. [7], Chynoweth [8], and Shutts [9,10] have constructed solutions by this method in a number of situations and obtained agreement with observed behaviour, lending physical plausibility to the solutions. In this paper we show that these solutions can be defined in a mathematically rigorous way, and existence and uniqueness proved. This is done by identifying the method of finding the trajectories with a generalised solution of the Monge-Ampere equation, and making use of the large body of theory available for that equation, [11]. We then show that the solutions can be interpreted as a sequence of minimum energy states. At each time instant the energy is minimised with respect to rearrangements of the fluid that conserve parcel properties, essentially entropy and a form of momentum which allows for the rotation of the coordinate system. This relies on recent results in rearrangement theory proved by Burton [12]. This characterisation gives a strong physical justification to the model.

2. DESCRIPTION OF SOLUTIONS

Consider the unforced semi-geostrophic equations in the form used by Hoskins [2].

$$Du_g/Dt + \partial\phi/\partial x - fv = 0 \tag{2.1}$$

$$Dv_g/Dt + \partial\phi/\partial y + fu = 0 \tag{2.2}$$

$$D\theta/Dt = 0 \tag{2.3}$$

$$(fv_g,\ -fu_g,\ g\theta/\theta_o) = \nabla\phi \tag{2.4}$$

$$D/Dt \equiv \partial/\partial t + \mathbf{u}.\nabla \tag{2.5}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2.6}$$

These equations are to be solved in a closed region Ω in $\mathbf{R}^3$ with $\mathbf{u}\cdot\mathbf{n}$ given on $\partial\Omega$. f is the Coriolis parameter, here assumed constant, g is the acceleration due to gravity, θ_o is a reference value of the potential temperature θ, and ϕ is the geopotential. Equation (2.4) includes the definition of geostrophic wind and the statement of hydrostatic balance. The velocity $\underline{u}$ has to be determined implicitly.

The structure of the solutions becomes clearer by making the substitutions used in [2] and [3] and Purser and Cullen [13]. Set $X=x+v_g/f$, $Y=y-u_g/f$, $Z=g\theta/f^2\theta_o$, $P=(\phi/f^2)+\frac{1}{2}(x^2+y^2)$. The equations then become:

$$DX/Dt = f(y-Y) \tag{2.7}$$

$$DY/Dt = f(X-x) \tag{2.8}$$

$$DZ/Dt = 0 \tag{2.9}$$

$$(X,Y,Z) = \nabla P. \tag{2.10}$$

The continuity equation (2.5) can be combined with the other equations as in [2] to give the inverse potential vorticity equation:

$$D\rho/Dt = 0, \tag{2.11}$$

where

$$\rho = \partial(x,y,z)/\partial(X,Y,Z). \tag{2.12}$$

It is sometimes more convenient to write this as

$$Dq/Dt = 0, \tag{2.13}$$

where $q=\rho^{-1}$. It is also convenient to define a dual potential $R(\underline{X})$ as

$$R(\mathbf{X}) = \mathbf{x}\cdot\mathbf{X} - P(\mathbf{x}(\mathbf{X})), \tag{2.14}$$

whence it is shown in [13] that

$$\nabla R = \mathbf{x}. \tag{2.15}$$

In cases where the solutions are sufficiently smooth

$$\rho = \det \{H(R)\}, \tag{2.16}$$

$$q = \det \{H(P)\}, \tag{2.17}$$

where H denotes the Hessian matrix whose components are $\partial^2/\partial x_i \partial x_j$.

This notation has been chosen to emphasise a geometrical interpretation of the problem in which the solution at any time consists of a transformation between physical space ($\mathbf{x}$ space) and data space ($\mathbf{X}$ space). As well as considering the solution conventionally as finding $\mathbf{X}(\mathbf{x})$ for $\mathbf{x}\in\Omega$, it can be considered as the dual problem of finding an $\mathbf{x}$ in Ω for each $\mathbf{X}$ for which $\rho(\mathbf{X})\neq 0$. The first stage in the solution is to solve this dual problem, for which the only information needed is $\rho(\mathbf{X})$. The Monge-Ampere equation (2.16) is then solved for R using the boundary condition that $\nabla R\in\partial\Omega$ as $\mathbf{X}\to\infty$, which is a standard boundary value problem for this equation.

The evolution equations can now be solved by reinterpreting equations (2·7) to (2·9) as defining a particle velocity field

$$\mathbf{U}=(f(y-Y),f(X-x),0) \tag{2·18}$$

in $\mathbf{X}$ space. Physical particles preserve their values of ρ because of (2·11), so the ρ equation in $\mathbf{X}$ space becomes the standard conservation law

$$\partial\rho/\partial t + \nabla_X\cdot(\rho\mathbf{U}) = 0. \tag{2·19}$$

This equation can be used in integral form for non-smooth solutions where ρ is a generalised function. It is then integrated forwards in time to give a new ρ distribution and the transformation between $\mathbf{x}$ space and $\mathbf{X}$ space is reconstructed.

This type of solution procedure can be applied to several of the approximate systems of equations used in meteorology and oceanography, as described by Hoskins et al. [14]. It also forms the natural extension of the coordinate transformation method of [4] to generalised solutions. The physical interpretation is that fluid parcels of specified volume and with specified 'absolute' momentum (X,Y) and buoyancy Z can be uniquely arranged in the fixed volume Ω so that a potential function P exists and the geostrophic and hydrostatic relations (2.10) are satisfied. It turns out that an additional dynamical stability condition, equivalent to parcel stability (Shutts and Cullen [15]) is needed to make the arrangement unique. The values of X and Y for fluid parcels change with time according to (2.7) and (2.8), and the parcels move in physical space in such a way as to satisfy (2.10). This motion is well-defined if the unique rearrangement result can be proved. In the unforced problem it can be shown that the required motion of particles is continuous and can be used to define a velocity field $\mathbf{u}$. This velocity field need not be smooth and, in particular, fluid can separate from the boundary though it cannot pass through it. If source terms are included in (2.7) to (2.9), fluid parcels may sometimes jump discontinuously to new positions. Under these conditions a $\mathbf{u}$ field cannot be defined, though the generalised solution still exists.

In subsequent sections we prove the existence and uniqueness of solutions to this problem under the additional condition that P is a convex function. This is the extra dynamical stability condition referred to above. It requires P to be continuous and differentiable almost everywhere but requires no other smoothness. The dual potential R is then also convex. The proof identifies the problem with a generalised solution of the Monge-Ampere equation at each time, combined with a solution of ordinary differential equations in time. Results proved in Pogorelov [11] for the Monge-Ampere equation are extended as necessary for this problem. Uniqueness of solutions to the ordinary differential equations can also be proved, though a redefinition is required in situations where the data are constant in regions of physical space. In that case ρ is an unbounded distribution and the right hand sides of equations (2·7) and (2·8) are multivalued. A weak form of the equations must then be used instead.

An important characterisation of these solutions, which greatly enhances their physical importance, is that they minimise the energy of the fluid at each time instant under rearrangements of the fluid which conserve parcel values of **X**. This is closely related to the Hamiltonian interpretation of these equations by Salmon [3] and Shutts [16]. The proof of this for generalised solutions is set out in a later section. The argument is illustrated here for smooth solutions.

The kinetic energy of the fluid is approximated by its geostrophic value. It is then shown in [2] that the total energy density E of a fluid parcel relative to its value in an isentropic basic state atmosphere can be written as

$$E = \tfrac{1}{2}(u_g^2+v_g^2) - g\theta z/\theta_0, \tag{2.20}$$

$$f^{-2}E = \tfrac{1}{2}(x^2+X^2+y^2+Y^2) - (xX+yY+zZ). \tag{2.21}$$

It can then be shown that equations (2.1) to (2.5) imply the energy equation

$$DE/Dt = -\mathbf{u}\cdot\nabla\phi. \tag{2.22}$$

Given a smooth volume-preserving rearrangement of the fluid, if δ denotes a change under the rearrangement, then $\delta\mathbf{x}$ can be written as proportional to $\nabla\times\mathbf{A}$ for some vector potential $\mathbf{A}$. The rearrangements conserve $\mathbf{X}$ and the integral of $\mathbf{x}^2$ is independent of the rearrangement. The resulting change in the total energy of the system deduced from (2.21) is then proportional to

$$\delta\int(-\mathbf{x}\cdot\mathbf{X}),$$

$$=-\int(-\delta\mathbf{x}\cdot\mathbf{X}),$$

$$=-\int(-\nabla\times\mathbf{A}\cdot\mathbf{X}).$$

$$=\int(\mathbf{A}\cdot\nabla\times\mathbf{X}) + \text{a boundary term}.$$

If the rearrangement is within a rigid boundary, the boundary term vanishes and the condition for the energy change to vanish for arbitrary displacements is that $\mathbf{X}=\nabla P$ for some P. This identifies the geostrophic and hydrostatic relations with the condition for the energy to be stationary. A rather longer argument set out in [15] shows that the condition for the energy to be a minimum is that P is convex.

3. Construction of the solution at a fixed time

Suppose that at some time t, ρ is given as a function of $\mathbf{X}$, subject to the condition that integrating ρ over all $\mathbf{X}$ gives the volume of the given region Ω in physical space. Then the remaining variables can be computed in principle by the following steps:

(i) Use the definitions of ρ and R to give

$$\det(\partial^2 R/\partial X_i \partial X_j) \equiv \partial(\nabla R)/\partial(\mathbf{X}) = \rho. \tag{3.1}$$

(ii) Equation (3.1) is a Monge-Ampere equation for R in terms of ρ. It must be solved given the boundary conditions that $\nabla R \equiv \mathbf{x}$ is always within Ω for all $\mathbf{X}$. The equation can also be given the geometric interpretation of finding a convex surface with given curvature.

(iii) The solution for $R(\mathbf{X})$ allows ∇R to be calculated, and hence the mapping $\mathbf{X}\to\mathbf{x}$ which assigns data values to points in physical space.

(iv) All information required to advance the solution of (2.6) - (2.10) in time is now available.

The values of the total velocity $\mathbf{u}$ in physical space do not have to be computed to advance the solution in time, but can be diagnosed by calculating $D\mathbf{x}/Dt$.

The key result required is that (3.1) can be solved with the given boundary conditions. In [6] it is solved using a piecewise constant approximation to the data. This is equivalent to specifying a discrete set of data values $\mathbf{X}_i$. Associated with each value is a ρ_i, which is the volume of fluid with this value. Provided the ρ_i add up to the volume of Ω, it is shown in [6] show that the fluid can be arranged uniquely in physical space so that geostrophic and hydrostatic balance, expressed by $\mathbf{X}=\nabla P$, is satisfied, and so that the fluid is stable to overturning motions, expressed by the fact that P is convex.

This finite dimensional result is proved as the first step in the theory of generalised solutions to the Monge-Ampere equation, Pogorelov [11]. The results are then extended to general data and allow the proof of existence and uniqueness of solutions to (2.13) to be completed. Though many of the results in [11] are stated in two-dimensions, they actually hold in any number of dimensions. Three-dimensional versions are required here. The geometrical interpretation of the problem requires construction of the hypersurface $P(x,y,z)$. This is a surface in four dimensional space, and we use s to denote the fourth coordinate.

Only the theorems are stated in this paper, since the proofs are trivial modifications of those given in [11].

Definition ([11], Chapter III, p.19)

Let V be an arbitrary polyhedral angle having convexity in the direction $s<0$, with vertex (X,Y,Z). The total curvature associated with the angle V is defined as the projection of the angle on **X** space

$$\lambda(V) = \iiint_{(V)} \lambda'(x,y,z)\,dxdydz, \tag{3.2}$$

where the integration is carried out over all planes of support of the angle V and $\lambda'(x,y,z)$ is the function $1/\surd(1+x^2+y^2+z^2)$. If, instead, λ' is an arbitrary positive continuous function of **x**, **X**, and s, and is nonincreasing with respect to s, the function λ defined by (3.2) is called the generalised curvature ∎

Theorem 1 ([11], Chapter III, Theorem 3a, p.23)

Let V be a polyhedral angle which projects onto the whole of (X,Y,Z) space, having convexity in the direction $s<0$, let $(g_1,\cdots,g_n)$ be lines parallel to the s axis, let $(\lambda_1,\cdots,\lambda_n)$ be positive numbers, let λ' be the function defined above. Suppose that

$$\iiint_{(V)} \lambda'(x,y,z)\,dxdydz = \textstyle\sum_k \lambda_k. \tag{3·3}$$

Then there exists an infinite convex polyhedron which projects in a single-valued manner into **X** space, having convexity in the direction $s<0$, with limit angle V, vertices on the lines g_k, and curvatures λ_k at these vertices.

This polyhedron is unique to within displacements in the direction of the s axis ∎

This construction will be used to give a surface $R(\mathbf{X})$. The prescribed limit cone will be chosen to satisfy $\nabla R \in \partial A$, where ∂A is the boundary of the given region in physical space. The curvatures λ_k will be given by values of ρ.

Theorem 2 ([11], Chapter IV, Theorem 3a, p.33)

Suppose given a completely additive set function $\rho(\mathbf{X})$ which is non-negative in the convex region Γ and zero in the exterior of this region. (A set function is a function which takes values on subsets, rather than just at individual points). Suppose V is a convex cone which projects onto the whole of **X** space, having convexity in the direction $s<0$.

Let $\lambda'(\mathbf{x})$ be the function defined above. Suppose that it satisfies the condition

$$\iiint_{(V)} \lambda'(x,y,z)\,dxdydz = \rho(\Gamma). \tag{3·4}$$

Then there exists an infinite convex surface R with limit cone V and curvature λ equal to ρ ∎

This theorem extends the geometrical construction used in [6] to general convex surfaces, and allows semi-geostrophic solutions to be constructed for general, not just piecewise constant, data. No results on uniqueness appear in [11] for this case. Uniqueness is proved there for the polyhedral case and the case with strictly positive curvature. The method is to suppose there are two different solutions R_1 and R_2. The arbitrary constant allows it to be assumed that $R_2-R_1 \geqslant 0$ with equality somewhere. If the two R surfaces then separate from each other, there must be some subset of Γ on which R_1 and R_2 subtend different solid angles, which contradicts the requirement that they have equal ρ.

In the general case needed here, the difficulty is that the solid angle subtended by a subset of Γ may not be well-defined because ρ may have singularities. This difficulty, however, only arises if ρ has an integral on the boundary of the chosen subset of Γ which is greater than some non-zero value ν. Since the integral of ρ over Γ is finite, there can only be a finite number of such disjoint boundary sets. An arbitrarily small shrinkage of the subset will then make the solid angle well-defined. The version of this argument needed for proof of continuous dependence of the solution on the data is given below.

Lemma 1

Any connected set B of **X** with finite measure contains a subset A of finite measure with the following properties. Given a neighbourhood $A(\varepsilon)$ of A with maximum distance ε from A, and a similar neighbourhood $A'(\varepsilon)$ of the complement A' of A in Γ, then $\rho(A(\varepsilon))\to\rho(A)$ as $\varepsilon\to 0$, and $\rho(A'(\varepsilon)')\to\rho(A)$ as $\varepsilon\to 0$.

Proof

A connected set A which does not have this property must have a boundary ∂A containing singularities of ρ whose integral over ∂A is greater than some $\nu>0$. Since the integral of ρ is finite, there can only be a finite number of such disjoint boundary sets within Γ and therefore within B, so there exists η with $0<\eta<1$ such that A can be shrunk by the factor η with respect to a point within A to give a new set ηA satisfying the conditions of the lemma ∎

Theorem 3

The convex surface R of Theorem 2 is unique up to an additive constant.

Proof

Theorem 2 ensures that such a R exists, call it R_1. It is clear from Theorem 2 that if R is a solution, so is R+c, where c is a constant. Suppose Theorem 3 is false, then there are solutions R_1 and R_2 with $R_1-R_2\neq c$. Without loss of generality, assume that R_2 is changed by a constant so that $R_2\geq R_1$ everywhere, and that there is at least one point

O with $R_1=R_2$. Note that this assumption uses the fact that R_1 and R_2 have the same limit cone.

Continuity of R_1 and R_2 and convexity of Γ ensures that $R_1=R_2$ on a closed subset C of Γ including O. If the theorem is false then R_2-R_1 must be strictly positive somewhere in Γ, and continuity of R_1 and R_2 means that this strict inequality must hold on an open subset of Γ with finite measure. There must thus be a point c with $\mathbf{X}=\mathbf{X}_0$ on the boundary of C, and a closed neighbourhood D of c in Γ in which R_2-R_1 is increasing in every direction away from c, and an open set of directions in which R_2-R_1 is strictly increasing for a finite distance. This neighbourhood must contain a neighbourhood A of c satisfying the conditions of Lemma 1. The total solid angle of R_2 within A is well defined and is the integral of all the gradients of R_2 within A which is therefore strictly less than that of R_1. This contradicts the requirement that both are equal to the integral of ρ over A ∎

Remark

It is not true that the curvatures of R_1 and R_2 at c have to be distinct to allow the surfaces to separate. A different form of this theorem which states that the function $\mathbf{X}(\mathbf{x})$ can be uniquely rearranged to form the subdifferential of a convex function has been claimed by Brenier [19].

These two theorems show that the solution of (3·1) can be uniquely constructed at any time. In order to extend this result to existence and uniqueness of a solution to the time-dependent problem (2·7) to (2·11), a form of continuous dependence of R on ρ is required. In order to solve the physical problem, continuity of R with regard to displacements of existing values of ρ is necessary, and any ρ satisfying the conditions of Theorem 2 must be allowed. Note that R is not continuous with respect to variations in ρ in the conventional sense, because a small perturbation to ρ could create negative values, and R could not then be calculated at all.

Definition

A displacement ρ' of ρ is defined by setting

$$\rho'(\mathbf{X}+\Delta(\mathbf{X}))=\rho(\mathbf{X}) \tag{3·5}$$

for all subsets of $\mathbf{X}$ space, where the mapping $\mathbf{X}\to\mathbf{X}+\Delta(\mathbf{X})$ maps Γ into itself with $\max|\Delta(\mathbf{X})|=\Delta$. The displacement mapping need not be volume preserving. Define a displacement R' of R by solving (3·1) with right hand side ρ'

Theorem 4

The convex function R' generated from R by a displacement as defined above tends to R as Δ tends to zero.

Proof

Let ρ' be the displacement of ρ. Then Lemma 1 means that any connected subset of Γ contains a set A of finite measure so that ρ' tends to ρ. Suppose that R' does not tend to R. Then for any δ however small there is an R'' with associated curvature ρ'' a finite distance ε from R generated by a displacement less than δ. As in the proof of Theorem 2, the arbitrary constants can be chosen so that $R''\geqslant R$ with equality on a closed subset C of Γ. In order for R'' and R to separate the required distance, there must be a neighbourhood A of some $c\in\partial C$ satisfying the conditions of Lemma 1 with $\rho''(A)$ different from $\rho(A)$ by $O(\varepsilon)$. Lemma 1 however requires this difference to be $O(\delta)$, which is a contradiction ∎

A stronger result can be proved if ρ is bounded. In this case a theorem quoted in [11,p.36] states that R is smooth, and hence continuously differentiable, [17,p.246].

Theorem 5

Under the conditions of theorem 4, if ρ is bounded and if the displacements conserve volume in **X** space then $\nabla R'$ tends to ∇R as Δ tends to zero.

Proof

The boundedness of ρ is preserved under such displacements, so the continuity of ∇R is preserved. Convergence of ∇R then follows from convergence of R, [17,p.248] ∎

This result is not true is ρ is unbounded because ∇R is then discontinuous.

4. Existence and uniqueness of solutions to the evolution equations

In this section we discuss the solution of the evolution equations (2·7) to (2·11), or the alternative dual space form (2·19). Given initial data $\rho(\mathbf{X})$ for all $\mathbf{X}$ with $\int\rho$ equal to the volume of Ω and with the support of ρ within a bounded set Γ in $\mathbf{X}$ space, calculate the velocity $\mathbf{U}$ using (2·18). This velocity is bounded because $\mathbf{x}\in\Omega$ and $\mathbf{X}\in\Gamma$, but is multivalued if ρ is a δ function. The velocity must also remain bounded for at least a finite time as the support of ρ can only expand at the velocity $\mathbf{U}$ which is initially bounded. The evolution equation (2·19) only makes sense if ρ and $\mathbf{U}$ are differentiable. Thus consider instead the integral form

$$\partial/\partial t(\rho(A)) = \int_{\partial A}\rho\mathbf{U}\cdot\mathbf{n}, \tag{4·1}$$

where A is any closed connected set of finite measure in $\mathbf{X}$ and the integral is taken round its boundary.

This equation can be solved by constructing a displacement mapping as in section 3, with

$$\Delta(\mathbf{X}) = \int\mathbf{U}(\mathbf{X})\,dt. \tag{4·2}$$

Define associated changes to ρ using (3·5). Equation (4·2) makes sense provided $\mathbf{U}$ can be calculated for all $\mathbf{X}$. This is possible if ρ is bounded.

ρ will remain bounded throughout a given time interval if it can be shown that the displacement rate $\mathbf{U}$ is non-divergent, for then ρ will be bounded by its initial values. If $\mathbf{U}$ is differentiable, a simple calculation based on (2·18) shows that it is non-divergent. Otherwise, for any set A as in equation (4·1), calculate $\int_{\partial A}\mathbf{U}\cdot\mathbf{n}$. Substituting for $\mathbf{U}$ shows that this is equal to

$$\int_{\partial A}(\partial R/\partial s-\mathbf{X}\cdot\mathbf{ds})ds$$

round the boundary of A, which vanishes.

Theorem 6

Given initial data $0 \leqslant \rho(\mathbf{X}) < \infty$ with bounded support Γ, equation (4·2) can be integrated forwards uniquely for a finite time interval.

Proof

The standard existence theorem for ordinary differential equations, e.g. [18,p.31], requires that the right hand side vary continuously with variations of the data. In this case the variations of the data are a displacement through a distance bounded by $U\Delta t$, where U is the maximum value of $\mathbf{U}$ and Δt is the time interval. The argument above shows that ρ remains bounded, so that Theorem 5 applies throughout the time interval. This theorem gives the necessary continuity ∎

In the more general case where ρ is unbounded but integrable $\mathbf{U}$ cannot be calculated everywhere. In this case there is a finite subregion Ξ of Ω with constant data values $\mathbf{X}$. Define $\mathbf{U}$ using the value of $\mathbf{x}$ at the centroid of this subregion:

$$\mathbf{U}(\mathbf{X}) = \int_{\Xi} (y-Y, X-x, 0). \qquad (4\cdot 3)$$

This definition agrees with (3·5) where $\mathbf{U}$ can be directly calculated. Now solve the evolution equation (4·2) using the displacement mapping (4·3), calculating ρ from (3·5), and calculating R from ρ.

Theorem 7

Given initial data as in Theorem 6 except that ρ need not be bounded above, form an evolution equation for R using the system (4·2), (4·3) and (3·5) as above. Write this equation as

$$\partial R/\partial t = f(R). \qquad (4\cdot 4)$$

Then equation (4·4) can be integrated forwards uniquely for a finite time interval.

Proof

Equation (4·3) ensures that the displacement (4·2) tends to zero as Δt tends to zero. The required continuity of R was established in Theorem 4. The rest of the argument is the same as for Theorem 6 ∎

In the smooth case treated in Theorem 6, individual particle trajectories can be followed and the equations integrated by a Lagrangian method. The positions of individual fluid particles within Ξ cannot be uniquely defined by these equations. Thus a Lagrangian method cannot be used. However, Theorem 7 shows that this does not prevent the determination of the data $\mathbf{X}$ as a function of $\mathbf{x}$. Physically this is because $\mathbf{X}$ is unaffected by a rearrangement of fluid within a region where the data is constant. Because trajectories cannot be calculated, the physical velocity $\mathbf{u}$ is undefined in Ξ. In reality, the velocity field in such regions may be quite chaotic, and cannot be approximated accurately using a simplified set of equations such as those used in this paper. It is a strength of the model that it still gives a well defined evolution of $\mathbf{X}(\mathbf{x})$ under these circumstances.

One of the desirable properties of the original form of the evolution equations is energy conservation. It is now shown that this is preserved when equation (4·3) is used. For simplicity, it is illustrated for the case when ρ is a set of delta functions ρ_i, corresponding to a polyhedral approximation to R. The general case can be recovered in the limit.

The total energy divided by f^2 is given as in (2·21) by

$$\int\left\{\tfrac{1}{2}(x^2+X^2+y^2+Y^2) - (xX+yY+zZ)\right\}\rho\, d\mathbf{X}, \tag{4·5}$$

where the integral is taken over data space. The terms x^2 and y^2 integrate to constants independent of the evolution. For polyhedral R with curvatures ρ_i at the vertices, the remaining terms reduce to

$$\sum\left\{\tfrac{1}{2}(X_i{}^2+Y_i{}^2) - (x_iX_i+y_iY_i+z_iZ_i)\right\}\rho_i. \tag{4·6}$$

The rate of change of the energy associated with each vertex is then

$$\{-Xd\bar{x}/dt-Yd\bar{y}/dt-Zd\bar{z}/dt-\bar{x}(\bar{y}-Y)+X(\bar{y}-Y)-\bar{y}(X-x)+Y(X-\bar{x})\}\rho, \tag{4·7}$$

where the subscript i has been omitted and $\bar{\mathbf{x}}$ is the centroid of the image Ξ of $\mathbf{X}$ in Ω. All terms except the first three cancel, and these take the form $\bar{\mathbf{u}}\cdot\nabla\phi$ as in (2·22).

To see that this sum vanishes, it is best to consider the evolution in physical space. The region Ω will be subdivided into polyhedral regions Ξ_i with volumes ρ_i and data values $\mathbf{X}_i$. Then the terms $d\bar{\mathbf{x}}_i/dt$ appearing in (4·7) are the mean velocity of points in Ω_i. A velocity field $d\mathbf{x}_i/dt$ can be defined over the whole of Ω_i by calculating the motion of its vertices and interpolating. This velocity field will be non-divergent because ρ_i is conserved and have mean value $\bar{\mathbf{x}}_i$. Since $\mathbf{X}$ is constant over Ω_i, the first three terms of (4·7) can be written

$$-\int_{\Omega i}\rho\mathbf{X}\cdot d\mathbf{x}/dt. \tag{4·8}$$

Since $\mathbf{X}=\nabla P$ and $d\mathbf{x}/dt$ is non-divergent, (4·8) represents the net flux of P into Ω_i. When summed over all i, and the zero flux condition on $\partial\Omega$ is imposed, the conservation of total energy is obtained.

The above analysis shows that the time dependent problem is well-posed and conserves total energy. It is then natural to ask if solutions exist indefinitely. The characteristic time scale in the model is the Earth's rotation period. Indefinite existence of solutions would make the model suitable for representing the evolution of the atmosphere or ocean over periods of several days or longer.

Theorem 8

Given initial data with bounded support Γ in $\mathbf{X}$, equation (4·4) has a bounded solution for all finite times.

Proof

The value of $|\mathbf{X}|$ following a particle obeys the equation

$$\begin{aligned} d|\mathbf{X}|/dt &= -\hat{\mathbf{Z}}\times(\mathbf{x}-\mathbf{X})\cdot\hat{\mathbf{X}} \\ &= -(\hat{\mathbf{Z}}\times\mathbf{x})\cdot\hat{\mathbf{X}}, \end{aligned} \tag{4·9}$$

where $\hat{\mathbf{X}},\hat{\mathbf{Z}}$ are unit vectors in the X and Z directions. Since $|\mathbf{x}|$ is bounded by the diameter of Ω, (4·9) shows that $\mathbf{X}$, and hence R, remains bounded for all finite times ∎

To obtain a stronger result, suppose that for some particles $|\mathbf{X}|$ becomes very large. Conservation of total energy means that the volume in physical space associated with these values scales as $|\mathbf{X}|^{-2}$. (4·9) shows that $|\mathbf{X}|$ continues to increase if the projection on the (X,Y) plane of $\mathbf{x}$ makes a positive angle with the projection of $\mathbf{X}$. Suppose that a small amount of the fluid can acquire almost all the total energy, so that it has a large $|\mathbf{X}|$. Conservation of Z means that it is X or Y which become large. Since Ω is convex and $\mathbf{X}$ is the gradient of a convex function on Ω, this region must adjoin the boundary of Ω near the point or segment where the direction $(Y,-X,0) = \hat{\mathbf{Z}}\times\mathbf{X}$ is tangent to it. Project Ω onto the (X,Y) plane and define a polar coordinate system (r,θ) in this plane with origin within Ω. The projection of the boundary of Ω is a convex closed curve whose boundary can be written $(r(\theta),\theta)$. Then the angle between $\mathbf{x}$ and $\mathbf{X}$ is positive if $r'(\theta)>0$ because then the angle of the tangent in direction $\mathbf{X}$ will be less than $\hat{\mathbf{Z}}\times\mathbf{X}$. The evolution equations (2·7-8) for an isolated large $\mathbf{X}$ indicates that it will rotate round the projection of Γ in the (X,Y) plane with period $2\pi f^{-1}$, and thus it will rotate round the boundary of Ω with the same period. Since the integral of $r'(\theta)$ round the boundary vanishes,it is likely that $|\mathbf{X}|$ tends to a constant as $t\to\infty$. It has not yet proved possible to make this argument completely rigorous.

5. The generalised solutions as minimum energy states

The results of the preceding two sections show that the model (2·1) to (2·6) has useful mathematical properties which make it a suitable candidate for describing the evolution of atmospheric flows containing discontinuities over long time periods, where the time scale is f^{-1}. In [1] and [2] the model is shown by scale analysis to be a good approximation to the Navier-Stokes equations if the flow is smooth and evolving on a time-scale greater than f^{-1}. When the solution becomes discontinuous, however, the assumptions behind the scale analysis are violated locally and the model cannot be justiofied in this way. In this section it is shown that the model represents the evolution of the fluid through a sequence of minimum energy states. Provided that this minimum energy state changes on a time scale long compared with that needed for the fluid to adjust to it, shown for instance in [15] to be less than f^{-1}, the model has strong physical plausibility even in the presence of discontinuities.

It is most convenient to work with the equations in physical space. The advection operator $\mathbf{u}\cdot\nabla$ is interpreted as carrying out a rearrangement of the fluid. Equations (2·7) to (2·9) then state that the data $\mathbf{X}$ evolves following fluid particles with tendency $(y-Y, X-x, 0)$, and that the fluid particles are then rearranged conserving $\mathbf{X}$ so that (2·10) is satisfied and P is convex. Under the conditions of Theorem 5, the trajectories are well defined and the rearrangement may be interpretable as advection by a velocity field $\mathbf{u}$. In general, however, the rearrangement can involve discontinuous changes in particle positions.

Define $\mathbf{X}(\mathbf{x})$ to be a rearrangement of $\mathbf{X}_0(\mathbf{x})$ if

$$\text{measure } \{\mathbf{x}\in\Omega \mid \mathbf{X} \geqslant \mathbf{c}\} = \text{measure } \{\mathbf{x}\in\Omega \mid \mathbf{X}_0 \geqslant \mathbf{c}\} \tag{5·1}$$

holds for all $\mathbf{c}\in\mathbf{R}^3$, where the inequalities are calculated component by component. Then the energy integral is given as an integral over physical space by

$$f^{-2}E = \int_\Omega \tfrac{1}{2}(x^2+X^2+y^2+Y^2) - (xX+yY+zZ)\,d\mathbf{x}. \qquad (5{\cdot}2)$$

Since the integral of the first bracket is conserved under rearrangements, the problem reduces to minimising the integral of $-\mathbf{x}{\cdot}\mathbf{X}$. Since under the assumptions of Theorem 2 both the data and the domain are bounded, this integral has a lower bound over the set H of rearrangements. Then if H is extended to its convex hull $\bar{H}$, there must be an element $h\in\bar{H}$ which minimises the integral, see [12] for the argument in the scalar **X** case, the vector case is a straightforward extension. It is more difficult to show that $h\in H$. Burton [12] shows that in the scalar case h must be an extreme point of the convex hull, and thus is actually a rearrangement. Again this result can be extended to the vector case.

Given a rearrangement h which minimises $-\mathbf{x}{\cdot}\mathbf{X}$, consider an interchange of fluid particles with data values $\mathbf{X}_i$ and initial positions $\mathbf{x}_i$. So that the new arrangement still fills Ω, this interchange must be cyclic, $\mathbf{x}_1\to\mathbf{x}_2\to\cdots\to\mathbf{x}_n$. If the fluid is in a minimum energy state, this interchange cannot decrease E, so that

$$\mathbf{X}_1\cdot(\mathbf{x}_2-\mathbf{x}_1) + \mathbf{X}_2\cdot(\mathbf{x}_3-\mathbf{x}_2) + \cdots + \mathbf{X}_n\cdot(\mathbf{x}_1-\mathbf{x}_n) \leq 0. \qquad (5{\cdot}3)$$

This is exactly the condition that the mapping $\mathbf{x}\to\mathbf{X}$ is cyclically monotone, and thus is the subdifferential mapping of a convex function on $\mathbf{x}$, [17,p.238]. The theorems of section 3 show that there is a unique convex function whose subdifferential is a rearrangement of any given $\mathbf{X}(\mathbf{x})$. This proves

Theorem 9

Given any bounded vector valued function $\mathbf{X}(\mathbf{x})$ on a convex domain Ω, there is a unique rearrangement $h\mathbf{X}$ of it which minimises the energy integral (5·2). The rearranged function $h\mathbf{X}$ is the subdifferential mapping of a convex function $P(\mathbf{x})$, and is the only rearrangement which can be so characterised.

6. Discussion

Generalised solutions of a standard simplified model of motions in the atmosphere and ocean have been constructed. It has been shown that these solutions make sense, by using the theory of the Monge-Ampere equation.The model can be integrated in time for long periods, possibly indefinitely, to describe the evolution of the atmosphere or ocean. The solutions conserve energy. They can be interpreted as following a sequence of minimum energy states of the system, where the energy is minimised under rearrangements of the fluid which conserve entropy and a form of momentum which allows for the rotation of the coordinate system. A real flow adjusts to such a minimum energy state take place on a time-scale less than f^{-1}, where f is the Coriolis parameter. The model is therefore physically plausible provided that the solution changes on a time scale longer than f^{-1}. In practice this requires that the wind direction and direction of the pressure gradient change slowly following the motion of a particle. Discontinuities can be formed in the solution. Near such discontinuities the simplified model will not be an accurate solutions of the equations of motion, and the scale analysis which relates the two will not apply.

The model provides a way of deriving a slowly varying subset of motions in the atmosphere and ocean. This has more usually been attempted by direct scale analysis, for instance by Kreiss [20]. The latter approach requires smoothness in the vertical and can only be argued rigorously for a time interval of the order f^{-1}. The present approach removes both these limitations at the cost of losing the rigorous link to the full equations of motion.

Knowledge of the existence of such a system is important in designing weather and climate models. The special properties which allow the theorems to be proved should be respected by numerical methods as far as possible, to allow accurate long term integrations. The application to numerical methods is discussed by Cullen [21].

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