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Numerical Weather Prediction

Generalised semi-geostrophic theory on a sphere

Forecasting Research Technical Report No. 400
Joint Centre for Mesoscale Meteorology Report No. 138

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Generalised semi-geostrophic theory on a sphere

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January 29, 2003

Abstract

It is shown that the solution of the semi-geostrophic equations for shallow water flow can be found and analysed in spherical geometry by methods similar to those used in the existing f-plane solutions. Stable states in geostrophic balance are identified as energy minimisers and a procedure for finding the minimisers is constructed, which is a form of potential vorticity inversion. This defines a generalisation of the geostrophic coordinate transformation used in the f-plane theory. The procedure is demonstrated in computations.

The evolution equations take a simple form in the transformed coordinates, though, as expected from previous work in the literature, they cannot be expressed exactly as geostrophic motion. The associated potential vorticity does not obey a Lagrangian conservation law, but it does obey a flux conservation law, with an associated circulation theorem.

The divergence of the flow in the transformed coordinates is primarily that naturally associated with geostrophic motion, with additional terms coming from the curvature of the sphere and extra 'curvature' resulting from the variable Coriolis parameter in the generalised coordinate transformation. These terms are estimated, and are found to be very small for normal data. The estimate is verified in computations, confirming the accuracy of the local f-plane approximation usually made with semi-geostrophic theory.

1 Introduction

Quasi-geostrophic theory has for a long time been the most widely used model of large-scale atmospheric circulations. This is because of its conceptual simplicity, and the possibility of finding analytic solutions. However, the quasi-geostrophic approximation in its standard form requires a constant Coriolis parameter and a fixed reference state static stability that is independent of horizontal position. Neither of these approximations is valid on large scales in the atmosphere, though that does not prevent the solutions from being conceptually useful. The geostrophic momentum approximation, originally introduced by Eliassen (1948), and developed and promoted by Hoskins (1975), allows the use of the correct variation of the Coriolis parameter and the static stability. In order to retain energetic consistency in the resulting equations, the geostrophic approximation is only made in the calculation of the momentum, not in the fluid trajectory. This particular feature of the approximation is now well understood in terms of hamiltonian mechanics. The resulting semi-geostrophic equations can describe a number of subsynoptic flows such

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as fronts embedded in cyclones, and interactions of large scale flow with topography and convection. These were reviewed by Cullen *et al.* (1987). However, explicit solutions of the equations have normally been obtained only with a constant Coriolis parameter, where the geostrophic coordinate transformation of Hoskins reduces the equations to a similar form to the quasi-geostrophic equations. The geometric solution procedure introduced by Cullen and Purser (1984) also relies on a constant Coriolis parameter. The solutions have proved conceptually useful despite this restriction.

The semi-geostrophic equations have, however, been integrated numerically on the sphere, without using a coordinate transformation. Mawson and Cullen (1992) show that ageostrophic cross-equatorial flows can be predicted as a response to suitably imposed forcing. Mawson (1996) shows, using a shallow water version of the equations, that the model supports the same Rossby wave solutions as the full equations, as long as the geostrophic wind satisfies the inertial stability condition. However, the inertial stability condition severely constrains the permitted solutions close to the equator. The semi-geostrophic approximation thus contains the 'weak temperature gradient' approximation which has recently become popular in tropical studies, e.g. Polvani and Sobel (2002). Cullen (2000) verifies that the errors in large-scale semi-geostrophic solutions on the sphere decrease as the square of (L_R/L), where L is the horizontal length scale and L_R the Rossby deformation radius. Schubert *et al.* (1991) show, using a zonally symmetric form of the model, that many aspects of the observed Hadley circulation can be simulated. These studies confirm the appropriateness of the semi-geostrophic model for large-scale flows.

The f-plane solution procedure gives several benefits. It allows explicit solution in simple cases. It also shows that solutions can be obtained by transport of a single scalar, the potential vorticity, followed by inversion of the potential vorticity to obtain the remainder of the variables. Hoskins *et al.* (1985) showed that this is a generic procedure applicable to a number of balanced models. In the semi-geostrophic case, Benamou and Brenier (1998) and Cullen and Gangbo (2001) have used this structure to prove that the equations can always be solved given certain hypotheses. Lagrangian conservation of potential vorticity also provides a strong dynamical constraint on the system, since the values of the potential vorticity at any time are bounded by the initial values.

Because of these mathematical and physical properties, a number of attempts have been made to generalise the semi-geostrophic solution procedure to spherical geometry. However, this has not yet been achieved without altering the equations in some way. Salmon (1985) defines a set of equations directly in transformed space which is hamiltonian and conserves potential vorticity. He shows that the equivalent equations in physical space are not the same as the semi-geostrophic equations, but that the change to the equations is within the error made by using the semi-geostrophic equations as an approximation to the primitive equations. Magnusdottir and Schubert (1991), and Purser (1999), both approximate the semi-geostrophic equations in a way that assumes that the flow is approximately zonal, and then show that the resulting equations can be solved by a coordinate transformation. Shutts (1989) constructs a hamiltonian semi-geostrophic system for the sphere by regarding the spherical shell as a subset of general three-dimensional space. This leads to the "planetary" semi-geostrophic system, which recognises that the axis of rotation of the Earth is the special direction, rather than the local vertical. However, this model does not reduce to the local f-plane model on small regions of the Earth's surface. In view of the success of local f-plane models, and the belief that the spherical geometry will not fundamentally alter their results, we seek a version of the solution procedure that can be applied to the unmodified semi-geostrophic equations in spherical geometry. The approach adopted in this paper is based on preserving the form of the equations of motion in physical lagrangian variables — with a variable Coriolis parameter — while generalising the coordinate transformation in such a way that the f-plane geostrophic momentum transformation is recoverable in the limit of a constant Coriolis parameter.

Cullen and Purser (1989) showed that the potential vorticity inversion procedure for f-plane semi-geostrophic theory could be interpreted as a minimisation of the energy under the constraint of given inverse potential vorticity, where the inverse potential vorticity is defined as the Jacobian of the mapping from geostrophic and isentropic coordinates to physical coordinates. Henceforward we use the term 'potential density' for the inverse potential vorticity. In this paper, we generalise Cullen and Purser's (1989) result to the shallow water case on the sphere. The method used was first introduced by Cullen and Douglas (1998). It consists of finding a coordinate transformation on the sphere that generalises the geostrophic coordinate transformation. This has the property that the condition that the energy is stationary under the constraint of given potential density is equivalent to geostrophic balance. Following Cullen and Purser (1989), we then *define* the potential density inversion procedure to be minimising the energy under this constraint. We show in section 3.3 that this gives a concavity condition on the height field which is equivalent to a local inertial stability condition. This condition is the same as the ellipticity condition required in the solution procedure of Mawson (1996). It is also the analogue of the convexity principle used by Cullen and Purser (1984) for the f-plane case. A theorem by McCann (2001) can then be used to show that the potential density can be uniquely inverted, subject to a regularisation of the problem at the equator. We demonstrate the 'potential density inversion' in a computation.

In section 4, we derive the time evolution equation in the new coordinates. The potential density is transported by a velocity in the transformed space which is in the same direction as the geostrophic velocity, but with magnitude modified by terms that result both from the curvature of the sphere itself and from the variable Coriolis parameter. The local mass conservation equation in physical space transforms to a circulation theorem, so that the integral of the potential density within any material circuit in transformed space is conserved. The divergence of the velocity in transformed space is dominated by the variations of the Coriolis parameter. As material circuits move towards the equator, they expand and the potential density decreases.

In section 4.3, we show that the equations can be discretised in time by solving the energy minimisation problem at each time-step, followed by integration of a pair of explicit ordinary differential equations. By regularising the transformation at the equator, we can use the local concavity condition to show that the procedure converges as the time-step is refined. The limit solution will be the solution of a regularised problem. Because the height field is very flat near the equator, we can then show that a well-defined solution to the original problem is obtained as the regularisation is removed.

We illustrate the time-dependent solutions in section 4.4. In particular, we show that the local f-plane approximation to the potential vorticity is almost conserved, to the extent that it is not clear whether the non-conservation is analytic or numerical. This is because the divergence of the transport velocity in transformed space can be almost exactly removed by a rescaling of the transformed sphere. The remaining terms are shown to be small for realistic velocities, such as those used in the computations. Thus a diagnostic based on a potential vorticity calculation in real space will still be useful.

2 Basic theory

In this section we introduce the theory on the sphere and review the issues to be resolved. The kinematics and dynamics of shallow water theory on a plane, and its semi-geostrophic approximation, are discussed by Roulstone and Sewell (1996, 1997), for example, and we adopt their notation in quoting some of the equations.

Let $\mathbf{r}$ denote the position vector of a generic point, from a fixed origin in 3-dimensional Euclidean space, on the surface of the sphere of radius a. An increment along the surface can be written in physical components as $\mathrm{d}\mathbf{r} = a\cos\phi \mathrm{d}\lambda \mathbf{i}_\lambda + a\mathrm{d}\phi \mathbf{i}_\phi = (a\cos\phi \mathrm{d}\lambda, a\mathrm{d}\phi)$, with orthogonal unit vectors $\mathbf{i}_\lambda$ and $\mathbf{i}_\phi$ parallel to the coordinate circles of increasing longitude λ and latitude ϕ respectively.

2.1 The semigeostrophic equations

The motion of a typical particle in shallow water theory on a sphere can be described by expressing the current eulerian coordinates of the particle on the surface of the sphere

$$\lambda = \lambda(\alpha, \beta, t), \qquad \phi = \phi(\alpha, \beta, t) \tag{1}$$

as functions, on the right, of the particle labels (or lagrangian coordinates) α, β and the time t, such that $\alpha = \lambda(\alpha, \beta, 0), \beta = \phi(\alpha, \beta, 0)$. Incompressibility requires that

$$\frac{h(\alpha, \beta, 0)}{h(\alpha, \beta, t)} = \cos\phi \frac{\partial(\lambda, \phi\cos\phi)}{\partial(\alpha, \beta)} \tag{2}$$

where h is the fluid depth at the particle position. We denote $\cos\phi\partial(\lambda, \phi\cos\phi)/\partial(\alpha, \beta)$ by j and assume $0 < j < \infty$. This makes available the inverse description $\alpha = \alpha(\lambda, \phi, t)$, $\beta = \beta(\lambda, \phi, t)$ of the motion (1), and allows us to transfer between lagrangian and eulerian descriptions whenever required. In particular we can express $h(\alpha, \beta, t)$ as a different function $h(\lambda, \phi, t)$. Within the fluid we will assume $h > 0$, though noting that semi-geostrophic theory can also describe situations where $h = 0$ over part of the domain, see Cullen and Purser (1989). Unless otherwise stated, we shall use the same letter to denote a function and its generic values, as just illustrated.

If the coordinates are in a frame of reference rotating with the Earth, the particle acceleration $(a(\cos\phi\ddot{\lambda} - \sin\phi\dot{\phi}\dot{\lambda}), a(\ddot{\phi} + (\dot{\lambda})^2 \sin\phi\cos\phi))$ has an additional term $(-fa\dot{\phi}, fa\cos\phi\dot{\lambda})$, where the Coriolis parameter $f = 2\Omega\sin\phi$ is the component of the angular velocity vector normal to the surface and Ω is the spin of the Earth. The superposed dots signify the lagrangian time derivatives, following the particle. (Some authors write $\mathrm{D}^n/\mathrm{D}t^n$ for these derivatives.) The continuity equation can be written as the time derivative of (2) following the particle,

$$\dot{h} + h\nabla.\dot{\mathbf{r}} = 0. \tag{3}$$

where $\dot{\mathbf{r}} = (a\cos\phi\dot{\lambda}, a\dot{\phi})$ is the particle velocity.

The shallow water momentum equations on the sphere can then be written, using the spherical polar eulerian coordinates defined above, as

$$a(\cos\phi\ddot{\lambda} - \sin\phi\dot{\phi}\dot{\lambda}) - fa\dot{\phi} + \frac{g}{a\cos\phi}\frac{\partial h}{\partial\lambda} = 0, \quad a\ddot{\phi} + (\dot{\lambda})^2\sin\phi\cos\phi + fa\cos\phi\dot{\lambda} + \frac{g}{a}\frac{\partial h}{\partial\phi} = 0. \tag{4}$$

The constant acceleration due to gravity is denoted g, and the depth at a fluid particle position has now been written as a function $h(\lambda, \phi, t)$. Define the geostrophic wind (u_g, v_g) to have local physical components

$$u_g = -\frac{g}{fa}\frac{\partial h}{\partial\phi}, \quad v_g = \frac{g}{fa\cos\phi}\frac{\partial h}{\partial\lambda}. \tag{5}$$

The geostrophic momentum approximation to (4) is then

$$\dot{u}_g - \dot{\lambda}v_g\sin\phi - fa\dot{\phi} + \frac{g}{a\cos\phi}\frac{\partial h}{\partial\lambda} = 0, \quad \dot{v}_g + \dot{\lambda}u_g\sin\phi + fa\cos\phi\dot{\lambda} + \frac{g}{a}\frac{\partial h}{\partial\phi} = 0. \tag{6}$$

Equations (3), (5) and (6) are the semi-geostrophic system to be solved, but at present there are no results establishing existence and uniqueness properties for these equations because the Coriolis parameter is a function of position. We seek to solve the equations either in a given domain D of the surface of the sphere, or on the whole sphere. In the first case, it is a basic assumption that particles cannot enter or leave D across the boundary.

2.2 Conservation of energy and potential vorticity

In f-plane semi-geostrophic theories it is easy to show that the total energy, which is the sum of a geostrophic kinetic energy and a potential energy, is conserved following the motion of the fluid particles. Furthermore, also in the case of f-plane theories, the potential vorticity is a lagrangian conserved quantity. These issues are discussed in some detail in Roulstone and Sewell (1997) and in McIntyre and Roulstone (2002); the latter paper also explains in some detail why the geostrophic flow, and not the actual particle motion, appears in the conserved quantities.

When the Coriolis parameter becomes a function of position as it is in equations (6), no form of potential vorticity conservation is known to exist (e.g. discussion of shallow water semi-geostrophic theory in Roulstone and Sewell (1996), §3), save by making the approximations discussed by Salmon (1985), Shutts (1989), and Magnusdottir and Schubert (1991). These approximations amount to altering the original equations (3), (5), (6). However, as we shall now demonstrate, one can establish a conservation law for the total energy on the assumption that the energy remains finite as one approaches the equator (i.e. when f vanishes).

The total geostrophic kinetic plus potential energy, associated with the geostrophic wind, is defined by

$$G = \int \left(\tfrac{1}{2}h(u_g^2 + v_g^2) + \tfrac{1}{2}gh^2\right)\mathrm{d}\Sigma, \tag{7}$$

where $\mathrm{d}\Sigma = a^2\cos\phi\mathrm{d}\lambda\mathrm{d}\phi$ is the area element of the sphere, and the integration is either over a simply connected domain D of the sphere, or over the whole sphere. In discussing boundary conditions we now assume that D is a finite closed region of the sphere possessing a boundary. Then G has the property that

$$\dot{G} = -\frac{1}{2}\int g\nabla.(h^2\dot{\mathbf{r}})\mathrm{d}\Sigma = -\frac{1}{2}\oint gh^2\mathbf{n}.\dot{\mathbf{r}}\mathrm{d}s \tag{8}$$

where ds is the line element along the boundary of D. The first equality in (8) is a consequence of equations (6) and continuity (one form of which is that $h\mathrm{d}\Sigma$ is constant), before any boundary conditions are used. Here $\mathbf{n}$ denotes the outward unit normal to the boundary of D. Thus $\mathbf{n}$ is tangential to the sphere. The second equality in (8) depends on a 'divergence theorem' on the sphere. The sphere over which the integration is performed has to be considered as being imbedded in a three-dimensional space, so that the form of the divergence theorem we require is subtly different from what is usually found in standard textbooks and we therefore furnish a proof of this result in the Appendix.

The foregoing equations, and (8) in particular, imply the following result.

Theorem 1 *$\dot{G} = 0$ when (6) with continuity holds within D, together either with $\mathbf{n}.\dot{\mathbf{r}} = 0$ on the boundary of D, or with integration carried out over the whole sphere whereupon $\int \nabla.(h^2\dot{\mathbf{r}})\mathrm{d}\Sigma = 0$. Thus G is conserved as an overall property of the semi-geostrophic flow, even in spherical geometry.*

The remainder of this paper is devoted to the presentation and discussion of a method for establishing the existence of a set of solutions of the semi-geostrophic equations (3), (5) and (6) on the sphere for which the conservation of energy and the evolution of potential vorticity are well defined.

2.3 Identification of geostrophic balance with a stationary energy state

We now show that solutions of the semi-geostrophic equations in spherical geometry are characterized by being minimum energy states, as in the f-plane case, in a sense to be made precise as follows. With any shallow motion of local depth h on the sphere, we can associate a vector field having physical components (u, v) (say). Thus the vector is $u\mathbf{i}_\lambda + v\mathbf{i}_\phi$. It can be thought of as a velocity, but it does not need to have that interpretation, which is therefore purely notional, to suggest possible physical consequences. By analogy with (7) we can then define a notional energy

$$E = \int \left(\tfrac{1}{2}(u^2+v^2) + \tfrac{1}{2}gh\right) h\mathrm{d}\Sigma. \tag{9}$$

This is a functional of u, v and h, regarded as functions of position over Σ, which has the following property.

Theorem 2 *The conditions for the integral E to be stationary with respect to variations satisfying continuity $\delta(h\mathrm{d}\Sigma) = 0$* via

$$\delta h = -h\nabla.\delta\mathbf{r} \tag{10}$$

in D, where D is (part of) the sphere, and

$$\delta u = fa\delta\phi + v\sin\phi\delta\lambda, \quad \delta v = -fa\cos\phi\delta\lambda - u\sin\phi\delta\lambda \tag{11}$$

together with

$$h\mathbf{n}.\delta\mathbf{r} = 0 \tag{12}$$

on the boundary of D as necessary, are that

$$u = u_g, \quad v = v_g. \tag{13}$$

The stationary value of E is G.

Proof The calculation is formally similar to that which delivers (8) above. Using (10) first, followed by the divergence theorem for (part of) the sphere, we obtain

$$\delta E = \int (u\delta u + v\delta v + g\delta\mathbf{r}.\nabla h)h\mathrm{d}\Sigma - \tfrac{1}{2}g\oint h^2\mathbf{n}.\delta\mathbf{r}\mathrm{d}s. \tag{14}$$

Using (11) and (5), with (12) when required, we obtain

$$\delta E = \int (fa\delta\phi(u-u_g) - fa\cos\phi\delta\lambda(v-v_g))h\mathrm{d}\Sigma. \tag{15}$$

Then, for E to be stationary with respect to arbitrary variations $\delta\phi, \delta\lambda$, we must require (13) to hold. □

The substance of the result is that E is stationary when the notional velocity $\mathbf{u} = (u, v)$ is equal to the geostrophic wind within the fluid, and when the boundary conditions are satisfied. The choice of variations in (11) represents the effect of a notional displacement in a rotational system where the effect of any pressure perturbation generated by the displacement is neglected, as comparison of (11) with (6) shows. The increments in (11) are definitions, and in (10) δh is deduced from $\delta\mathbf{r}$ using incompressibility.

Shutts and Cullen (1987) analyse the physical significance of E being minimised, rather than just stationary, for the case of constant f. They show that it corresponds to the stability of a geostrophic state, viewed as a solution of the full primitive equations, to perturbations of the form

$$\delta u = f\delta y, \quad \delta v = -f\delta x, \quad \nabla\cdot(\delta x, \delta y) = 0 \tag{16}$$

which are the analogues of (11) in plane geometry. They also discuss the validity of the assumption that pressure perturbations can be neglected. They show (pp. 1321-1323) that it is valid if the basic flow and perturbations both satisfy the assumptions of semi-geostrophic theory, i.e. that one horizontal length scale is large.

We therefore derive necessary conditions for E to be minimised under the variations (11), closely following the method of Shutts and Cullen (1987). Rewrite (15), using $\delta\mathbf{r} = (a\cos\phi\delta\lambda, a\delta\phi)$, as

$$\delta E = \int \delta\mathbf{r}\cdot(-fa(v-v_g), fa(u-u_g))h\mathrm{d}\Sigma. \tag{17}$$

Then, taking a second variation,

$$\delta^2 E = \int \delta\left(fa\delta\mathbf{r}\cdot(-(v-v_g), u-u_g)\right)h\mathrm{d}\Sigma, \tag{18}$$

and since $\mathbf{u} = \mathbf{u}_g$ when $\delta E = 0$, this reduces to

$$\delta^2 E = \int fa\delta\mathbf{r}\cdot(-\delta(v-v_g), \delta(u-u_g))h\mathrm{d}\Sigma. \tag{19}$$

Substituting for $\delta\mathbf{u}$ from (11) and using $\mathbf{u} = \mathbf{u}_g$ gives

$$\delta^2 E = \int fa\delta\mathbf{r}\cdot\left((fa\cos\phi\delta\lambda + u_g\sin\phi\delta\lambda, fa\delta\phi + v_g\sin\phi\delta\lambda) + \delta(v_g, -u_g)\right)h\mathrm{d}\Sigma. \tag{20}$$

Equation (5) gives $(v_g, -u_g) = gf^{-1}\nabla h$. Thus $\delta(v_g, -u_g) = g\delta(f^{-1}\nabla h)$. If we write ∂ for a change at a fixed position in space caused by a displacement, then

$$g\delta(f^{-1}\nabla h) = g\partial(f^{-1}\nabla h) + g\delta\mathbf{r}\cdot\nabla(f^{-1}\nabla h) = g\partial(f^{-1}\nabla h) + \delta\mathbf{r}\cdot\nabla(v_g, -u_g) \tag{21}$$

Since $\partial f = 0$ and (10) implies that $\partial h = -\nabla \cdot (h\delta\mathbf{r})$, we have

$$g\delta(f^{-1}\nabla h) = -gf^{-1}\nabla(\nabla \cdot (h\delta\mathbf{r})) + \delta\mathbf{r} \cdot \nabla(v_g, -u_g). \tag{22}$$

Substituting (22) into (20) and integrating by parts gives

$$\begin{aligned}\delta^2 E = \int fa\delta\mathbf{r} \cdot ((fa\cos\phi\delta\lambda + u\sin\phi\delta\lambda, fa\delta\phi + v\sin\phi\delta\lambda) + \\ a(\cos\phi\delta\lambda, \delta\phi) \cdot \nabla(v_g, -u_g))h + ga\,(\nabla \cdot (h\delta\mathbf{r}))^2\,\mathrm{d}\Sigma.\end{aligned} \tag{23}$$

The second term is positive definite. The necessary condition for the energy to be minimised is therefore that the first term is positive definite. Writing it in the form $\delta\mathbf{r} \cdot \mathbf{P} \cdot \delta\mathbf{r}$, the condition is that the matrix

$$\mathbf{P} = f\begin{pmatrix} f + \dfrac{1}{a\cos\phi}\dfrac{\partial v_g}{\partial\lambda} + \dfrac{u_g\tan\phi}{a} & \dfrac{\partial v_g}{\partial\phi} \\ \dfrac{1}{a\cos\phi}\dfrac{\partial u_g}{\partial\lambda} - \dfrac{v_g\tan\phi}{a} & f - \dfrac{1}{a}\dfrac{\partial u_g}{\partial\phi}\end{pmatrix} \tag{24}$$

is positive definite.

We can see that this is the standard semi-geostrophic form of the inertial stability condition found by Shutts and Cullen (1987) using the local value of f and written in spherical polar coordinates. The derivatives of f do not enter the condition. The change of sign of f does not cause a problem, but the condition is very restrictive near the equator.

2.4 Solution procedure in physical space

We now describe the method used to solve the semi-geostrophic equations on the sphere in physical space. The numerical model of Mawson (1996) is based on this procedure. Assume we are given initial data h defined over the whole sphere or over some bounded subset D of it. Calculate the geostrophic wind from h using (5), and call the result $(u_g(0), v_g(0))$. Assume that this data are such that the associated geostrophic energy $G(0)$ calculated using (7) is finite, and calculate $\mathbf{P}$ using $\mathbf{u}_g(0)$ in (24). We can then write equations (3) and (6), following Schubert (1985), as:

$$\begin{aligned}\mathbf{P}\dot{\mathbf{r}} + \frac{\partial}{\partial t}\nabla h = -f^2\mathbf{u}_g \\ \frac{\partial h}{\partial t} + \nabla \cdot (h\dot{\mathbf{r}}) = 0.\end{aligned} \tag{25}$$

These equations can be combined to give

$$\frac{\partial h}{\partial t} - \nabla \cdot \left(h\mathbf{P}^{-1}\frac{\partial}{\partial t}\nabla h\right) = \nabla \cdot (h\mathbf{P}^{-1}f^2\mathbf{u}_g). \tag{26}$$

Equation (26) is a Helmholz equation for $\frac{\partial h}{\partial t}$ provided that the matrix $\mathbf{P}$ is definite. If $\mathbf{P}$ is positive definite, the eigenvalues of the principal part of the Helmholz operator will all be positive, and we can expect (26) to have a unique solution. Positive definiteness of $\mathbf{P}$ is exactly the energy minimisation condition derived in the previous subsection. In the f-plane case det$\mathbf{P}$ is the potential vorticity, and thus a constant of the motion, so that positive-definiteness will be preserved. We have to prove an equivalent condition in the spherical case.

We can interpret the solution procedure as follows. Given initial data $h(0), \mathbf{u}_g(0)$ as above, for some time step δt, calculate a notional velocity $\mathbf{u}$ with components

$$\begin{aligned} u &= u_g(0)\cos(f\delta t) - v_g(0)\sin(f\delta t) \\ v &= v_g(0)\cos(f\delta t) + u_g(0)\sin(f\delta t). \end{aligned} \tag{27}$$

Calculate a notional energy E by using this velocity in (9). It is easy to see that $|\mathbf{u}| = |\mathbf{u}_g|$ and thus $E = G(0)$. In general, however, (u, v) will no longer satisfy the geostrophic relation or the inertial stability condition. Next seek a displacement satisfying (11) and (10) to minimise the energy in the sense of Theorem 2, thus enforcing the geostrophic relation and inertial stability. Assuming that this can be found uniquely in the form of a displacement $\delta\mathbf{r}$, this determines a new geostrophic wind $(u_g(\delta t), v_g(\delta t)) = \mathbf{u} + \delta\mathbf{u}$ together with a true velocity $\dot{\mathbf{r}} = \delta\mathbf{r}/\delta t$. Theorem 2, as it stands, does not give the necessary information about uniqueness. Equation (26) suggests that there will be a unique solution if $\mathbf{P}$ is positive definite. The main purpose of the paper is to prove this.

2.5 Solution of the energy minimisation problem in the f-plane case

In the f-plane case, Cullen and Purser (1989) showed that the problem of minimising E subject to the variations (16) could be solved uniquely. They gave an intuitive proof of this, which has since been made rigorous by Douglas (1998) for the incompressible, three-dimensional stratified semi-geostrophic flows with rigid boundaries, and by Cullen and Gangbo (2001) for shallow water semi-geostrophic flow in a bounded region. In both cases the boundary conditions were that no fluid enters or leaves the region across the boundaries. We describe the procedure in a region D using cartesian coordinates (x, y). A key step is to rewrite the notional velocity as

$$(u, v) = f(y - Y', X' - x) \tag{28}$$

in terms of a new pair of generic cartesian coordinates X', Y'. The class of variations (16) and (10) under which the energy is to be minimised now take the form

$$\begin{aligned} \delta X' = \delta Y' &= 0 \\ \delta h &= -h\nabla.\delta\mathbf{r} \end{aligned} \tag{29}$$

These together imply that

$$\delta\sigma = 0 \tag{30}$$

where

$$\sigma = h\frac{\partial(x, y)}{\partial(X', Y')} \tag{31}$$

A state of rest with h equal to a uniform value h_0 corresponds to $\sigma = h_0$. Any other choice of σ implies some excess energy above the rest state.

Using (28), the notional kinetic energy term in (9) can be written as

$$\frac{1}{2}f^2 \int \left((X' - x)^2 + (Y' - y)^2\right) h\mathrm{d}x\mathrm{d}y, \tag{32}$$

which may thus be regarded as a weighted integrated distance between the physical position $\mathbf{x}$ and another associated point $\mathbf{X}' = (X', Y')$ in the euclidean space. It is therefore minimised by making the $\mathbf{X}'$ points correspond as closely as possible to the physical positions. This is the key standpoint from which our generalisations will follow.

If we define a distance $d(\mathbf{x}, \mathbf{X}')$ between $\mathbf{x}$ and $\mathbf{X}'$ in the f-plane to be such that its square is

$$d(\mathbf{x}, \mathbf{X}')^2 = f^2 \left((X' - x)^2 + (Y' - y)^2 \right), \tag{33}$$

the notional energy (9) can be rewritten

$$E = \int \left(\frac{1}{2} d(\mathbf{x}, \mathbf{X}')^2 + \frac{1}{2} g h \right) h \mathrm{d}x \mathrm{d}y. \tag{34}$$

Theorem 3 *Conditions for E in this form, and with constant f, to be stationary with respect to variations satisfying (29) within the domain D of the f-plane, and $\mathbf{n}.\delta\mathbf{r} = 0$ on the boundary, are that $\mathbf{u} = \mathbf{u}_g$.*

Proof The proof is just a rephrasing of Theorem 2 in cartesian coordinates, using (15) in particular, and the definitions of X' and Y' in (28). □

If we use (X, Y) to denote the stationary values of (X', Y'), the cartesian version of the definitions (5) reappear as definitions

$$X = x + \frac{g}{f^2} \frac{\partial h}{\partial x}, \quad Y = y + \frac{g}{f^2} \frac{\partial h}{\partial y} \tag{35}$$

in the present example of constant f. These X and Y are the geostrophic coordinates of Hoskins (1975) for shallow water theory. They are called geostrophic coordinates because, when f is a constant, one can show that the equations for momentum balance in cartesian coordinates can be re-written in the form $\dot{X} = u_g$, $\dot{Y} = v_g$.

The proof that E can be uniquely minimised is then carried out by showing that, given σ as a function of coordinates (X', Y'), there is a unique mapping from the coordinates (X', Y') to the physical coordinates (x, y) that minimises E and satisfies (31). The condition that no fluid can enter or leave D across the boundaries is enforced by requiring this mapping to be from $\mathbb{R}^2$ into D. In section 3, by writing the class of variations in the form (30), we will show that there is a unique minimiser of E in the spherical case also.

2.6 Variable Coriolis parameter

The transformation to the coordinates (X, Y) defined in (35) as functions of x and y, is the starting point for the application of the theory of lift and contact transformations in semi-geostrophic theory (Sewell and Roulstone 1994; Roulstone and Sewell 1997). This transformation theory describes the mapping $(x, y) \mapsto (X, Y)$, and its inverse, in terms of Legendre duality (Chynoweth and Sewell 1989). Hoskins (1975) showed that, as a consequence of the transformation properties of the equations on an f-plane, geostrophic motion in the new variables, $\dot{X} = u_g, \dot{Y} = v_g$, can be expressed entirely in terms of a stream function in the (X, Y)-coordinates and consequently the flow in these new coordinates is solenoidal. Roulstone and Sewell (1997, §7) broached the issue as to whether the f-plane semi-geostrophic equations could be generalised to variable-f geometries, while retaining the transformation properties of the f-plane equations, by modifying the mapping $(x, y) \mapsto (X, Y)$. They showed that the planetary semi-geostrophic equations of Shutts (1989) can be obtained, following such a strategy, by introducing a pair of 'intermediate' variables (x_1, x_2) that are related to (x, y) via

$$x_1 = f_0 x, \quad x_2 = \int_{y_0}^{y} f(u)\mathrm{d}u \tag{36}$$

where f_0 and y_0 are given constant (datum) values of f and y. By considering a mapping $(x_1, x_2, \psi) \mapsto (X_1, X_2, \Psi)$ defined by

$$\left.\begin{array}{c} X_1 = x_1 + \dfrac{\partial\psi}{\partial x_1}, \quad X_2 = x_2 + \dfrac{\partial\psi}{\partial x_2} \\ \Psi = \psi + \frac{1}{2}\left(\left(\dfrac{\partial\psi}{\partial x_1}\right)^2 + \left(\dfrac{\partial\psi}{\partial x_2}\right)^2\right), \ \dfrac{\partial\Psi}{\partial X_1} = \dfrac{\partial\psi}{\partial x_1}, \ \dfrac{\partial\Psi}{\partial X_2} = \dfrac{\partial\psi}{\partial x_2}, \end{array}\right\} \tag{37}$$

then, if $\psi(x_1, x_2, t) = gh(x, y, t)$, it is straightforward to prove that the pseudo-hamiltonian equations for solenoidal flow in (X_1, X_2)-space

$$\dot{X}_1 = -f_0\frac{\partial\Psi}{\partial X_2}, \quad \dot{X}_2 = f_0\frac{\partial\Psi}{\partial X_1} \tag{38}$$

transform under (37) to

$$\dot{u}_g - f(y)\dot{y} + g\frac{\partial h}{\partial x} = 0, \quad \frac{f(y)}{f_0}\overline{\left(\frac{f(y)v_g}{f_0}\right)}^{\cdot} + f(y)\dot{x} + g\frac{\partial h}{\partial y} = 0, \tag{39}$$

which are representations of the zonal and meridional planetary semi-geostrophic equations. (Note that, in the above, the components of the geostrophic wind, (u_g, v_g), are defined in terms of a variable Coriolis parameter.)

This result prompted an investigation into whether there exists an intermediate pair of coordinates, similar to those defined above, which would enable equations of the form (38) to be transformed into semi-geostrophic equations of the type given by (6). It emerged that the required mapping from (x, y) to (x_1, x_2) would not be integrable, but it was suggested that such difficulties might be circumvented by introducing a non-holonomic transformation that would take the differential form

$$\mathrm{d}x_1 = f(y)\mathrm{d}x. \tag{40}$$

However, such suggestions were not made explicit.

Later in this paper we introduce a transformation (defined by (62)), subtly different from (36) and (40), which enables us to explicitly transform the equations (6) into a form similar to (38), but with non-constant coefficients and therefore representing a compressible flow in the new coordinates. This new transformation is based on interpreting (33) as a metric, or distance function, on phase space, with a scale factor given by the Coriolis parameter. Therefore, in a certain sense, a new transformation is to be defined by rescaling a metric and not by rescaling individual coordinates as in (36) and (40). The generalisation of this idea to variable-f geometries, and the construction of a new coordinate transformation, is the subject of the next section.

3 The geostrophic momentum transformation on the sphere

3.1 Calculation of a coordinate transformation by energy minimisation

In this section, we use the model of Mawson (1996) to illustrate that the energy defined in Theorem 2 can be minimised in the spherical case, subject to the variations (10, 11);

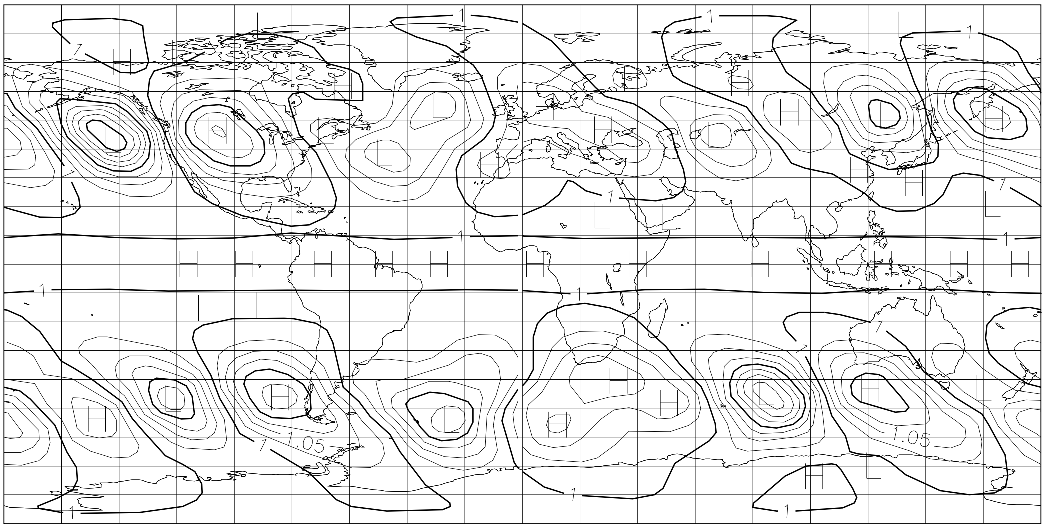


Figure 1: Initial distribution of the dimensionless quantity σ/h_0 for equation (43). Contour interval 0.025.

and that the solution can be expressed in the form of a coordinate transformation as in the f-plane case. This would require that the class of variations (10, 11) can be written in the form (30) for some choice of notional coordinates analogous to X', Y', so that there are functions Λ', Φ' of the spherical coordinates λ, ϕ and the notional velocity u, v such that (11) implies $\delta\Lambda' = \delta\Phi' = 0$ and that

$$(u, v) = 0 \Leftrightarrow (\Lambda', \Phi') = (\lambda, \phi). \tag{41}$$

Under these assumptions, the variations (10, 11) imply that

$$\begin{aligned} \delta\sigma &= 0, \\ \sigma &= h\frac{\partial(\lambda, \phi)}{\partial(\Lambda', \Phi')}\frac{\cos\phi}{\cos\Phi}. \end{aligned} \tag{42}$$

We use this to construct a coordinate transformation on the sphere, which can also be interpreted as a method of 'potential vorticity inversion' . Given $\sigma(\Lambda', \Phi')$, set

$$\begin{aligned} h(\Lambda', \Phi') &= \sigma h_0 \\ u &= 0 \\ v &= 0 \end{aligned} \tag{43}$$

As before, the choice $\sigma/h_0 = 1$ represents a trivial state of balance with no flow. An example of a non-trivial choice is shown in Fig. 1. This is designed to reproduce disturbances typical of a low-level atmospheric pressure field with geostrophic winds of about $15\mathrm{ms}^{-1}$.

We now seek a displacement $\delta\mathbf{r}$ so that the energy is minimised under (10, 11). If the displacement takes each point (Λ', Φ') to a point (λ, ϕ), then conservation of mass as expressed by (10) implies that

$$\sigma = h\frac{\partial(\lambda,\phi)}{\partial(\Lambda',\Phi')}\frac{\cos\phi}{\cos\Phi} \tag{44}$$

as required. There is an immediate difficulty with this procedure, as shown by Roulstone and Sewell (1997), eq. (7.20). Since the displacement is now over a finite distance, rather than infinitesimal, the variations of f in space mean that the displacements are non-integrable and the displacement suggested above does not give a well-defined answer. For instance, given $u = v = 0$ at $(0,0)$, displace a particle at that position to the point $(\pi/4, \pi/4)$ and calculate the change in $\mathbf{u}$ according to (11). If the displacement proceeds via the point $(0, \pi/4)$, the result is $(\Omega a, -\Omega a(1+\pi/4))$. If it is via $(\pi/4, 0))$, the result is $(\Omega a\sqrt{2}, 0)$.

To make the procedure well-defined, we must require a specific path for the displacement. The natural choice is a 'steepest descent' path in energy. Thus we minimise the energy (9) iteratively by calculating a displacement

$$\begin{aligned}\delta\mathbf{r} = &\quad \alpha(fv - \tfrac{1}{a\cos\phi}\tfrac{\partial h}{\partial\lambda}, -fu - \tfrac{1}{a}\tfrac{\partial h}{\partial\phi}) \\ = &\quad \alpha\left(f(v - v_g), -f(u - u_g)\right),\end{aligned} \tag{45}$$

where $\mathbf{u}_g$ is calculated from h using (5), and using (10) and (11) to update h, u, v. Substituting the second equation of (45) into (11) gives that

$$u\delta u + v\delta v = -\alpha f^2\left(u(u-u_g) + v(v-v_g)\right). \tag{46}$$

Then using (5), (14) and (45); and assuming no displacements across any boundary, we obtain

$$\delta E = -\alpha\int\left((u-u_g)^2 + (v-v_g)^2\right)h\mathrm{d}\Sigma \tag{47}$$

This is negative definite and vanishes when $\mathbf{u} = \mathbf{u}_g$. The solution defines the map from (Λ, Φ) to (λ, ϕ), and hence the coordinate transformation, by summing the displacements over all the iterations to give $(\delta\lambda_t, \delta\phi_t)$, and setting $\lambda = \Lambda + \delta\lambda_t, \phi = \Phi + \delta\phi_t$. By construction, this map satisfies (44). However, we cannot guarantee that this procedure will converge. If it converges, it will be to a stationary point of the energy, because of Theorem 2. However, we cannot guarantee at this stage that it is a minimum, much less a global minimum. These issues will be addressed in section 3.3.

The result of applying the procedure to the first guess field shown in Fig. 1 is illustrated in Fig. 2., 100 iterations were used. The semi-geostrophic shallow water model of Mawson (1996) was used. The initialisation procedure described there (p.276: initialisation stage 2) is equivalent to using (45). The 'correction velocity' U_A defined on p.271 of that paper generates the displacement required by (45), and the updates using (10) and (11) are equivalent to equations (10) and (9) in Mawson (1996). Fig. 2 shows that positive anomalies in σ/h_0 become positive anomalies of h. The h field is smoother than the σ field. This is to be expected, since σ is related to the potential vorticity, which is expected to have smaller scales than the depth field.

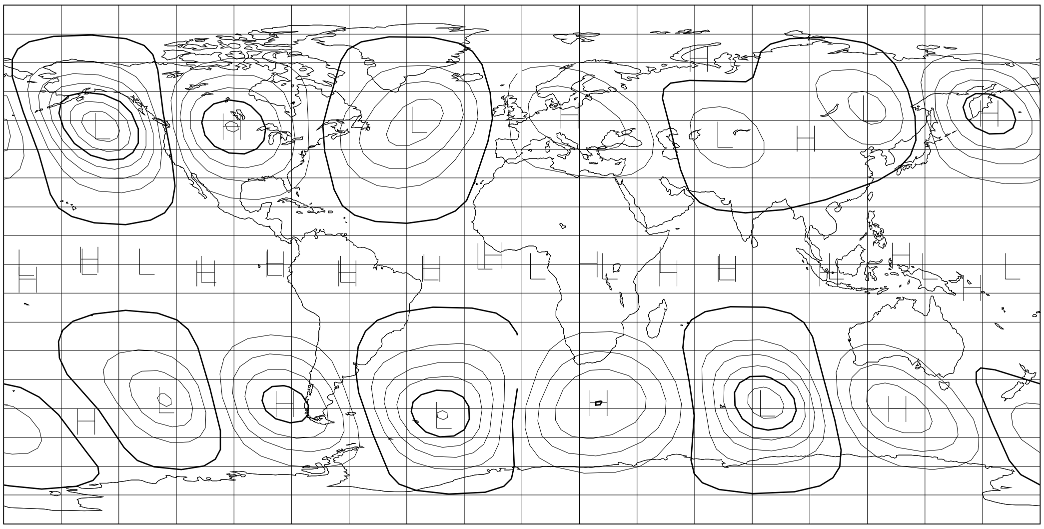


Figure 2: Distribution of h derived from initial field shown in Figure 1. Contour interval $250m$. Bold contours at 10600, 10700, 10800 m.

As a further check, the procedure can be reversed. Given initial data with depth h and initial winds $u = u_g, v = v_g$, choose $\delta\mathbf{r}$ as minus the value given by (45), and iterate to a state where $u = v = 0$. Set σ equal to the final value of h. This procedure acts as a diagnosis of potential density from a given geostrophically balanced state. Since the initial magnitude of $\delta\mathbf{r}$ given by (45) will be zero, we must use a semi-implicit procedure of the form

$$\begin{aligned} \delta\mathbf{r}_n = & - \tfrac{1}{2}\alpha\left(fv - \tfrac{1}{a\cos\phi}\tfrac{\partial h}{\partial\lambda}, -fu - \tfrac{1}{a}\tfrac{\partial h}{\partial\phi}\right)_{n-1} \\ & - \tfrac{1}{2}\alpha\left(fv - \tfrac{1}{a\cos\phi}\tfrac{\partial h}{\partial\lambda}, -fu - \tfrac{1}{a}\tfrac{\partial h}{\partial\phi}\right)_{n}, \end{aligned} \tag{48}$$

where (10) and (11) with $\delta\mathbf{r} = \delta\mathbf{r}_n$ are used to update h, u, v from values at iteration level $n-1$ to values at iteration level n. The final values of h will be equal to the original σ, subject to numerical error. The top panel of Fig. 3 illustrates the final field. It is almost identical to the original field shown in Fig. 1. The lower panels show values of (Λ, Φ) calculated from (λ, ϕ) by summing the displacements defined in (48) over all the iterations. For the data chosen, the displacements are quite small, and the displacement of the latitude and longitude grid lines is only just visible. (Λ, Φ) can be regarded as the natural generalisation of geostrophic coordinates on the sphere; the coordinate transformation is not that far from the identity for these data.

The maximum and minimum values of h are also set out in Table 1. They show that the calculation of h from σ has been reversed to within about 1%. Tests with reducing α and increasing the number of iterations show convergence of the error to zero. The errors come both from the early relatively large iteration steps and from accumulated numerical

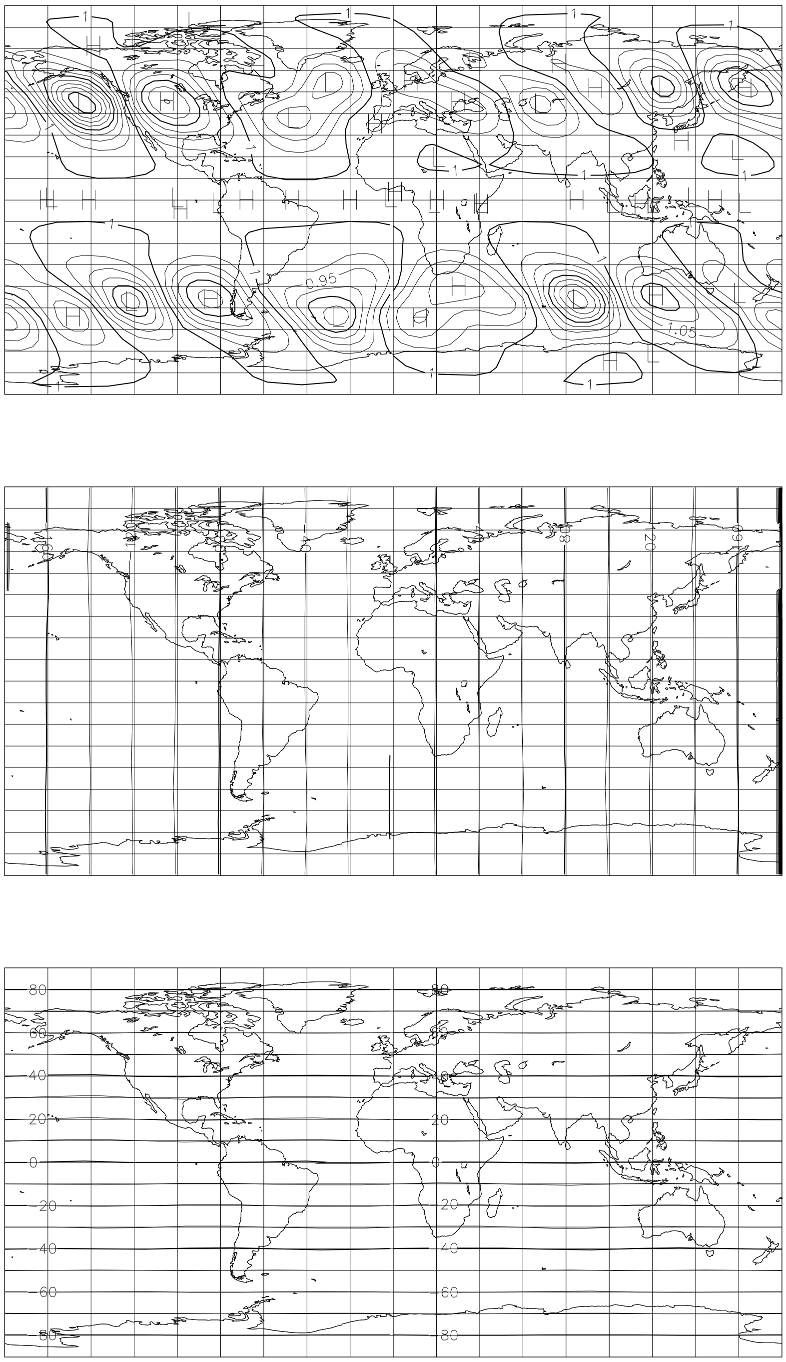


Figure 3: Top: Final distribution of h/h_0. Contour interval 0.025. Middle. Λ plotted as a function of λ, ϕ, contours every 20°. Bottom. Φ plotted against λ, ϕ. Contours every 10°.

error over all the iterations. Finite iteration steps only exactly follow the steepest descent path in the limit $\alpha \to 0$.

Table 1: Test of potential density inversion procedure.

Initial σ	Minimising h	Final σ
12324	10826	12313
8384	10533	8358

3.2 Generalised definitions of distance on the sphere

The starting point for defining the geostrophic momentum transformation rigorously on the sphere is to write the kinetic energy in terms of a distance function, as in (34). We start by considering how to measure the lengths of paths on the surface of the sphere. In the following we simplify the notation by writing $\mathbf{X}$ (in place of the $\mathbf{X}'$ used above) as the *generic* second point acting as the end point of a path which starts at the point $\mathbf{x}$ on the sphere. Particular solution values of such $\mathbf{X}$, like (35) above, will be identified in the text when they are needed without necessarily introducing fresh notation.

Let $\mathbf{r}$ be the position vector, from a fixed origin O in three-dimensional euclidean space, to a generic point on the surface of the sphere. Let a function $\mathbf{r}(s)$ of the distance s define a path $\mathbf{r} = \mathbf{r}(s)$ on the surface. Confine attention to part of the path of length l, so that $0 \leq s \leq l$, between end points

$$\mathbf{r}(0) = \mathbf{x} \ \ \text{(say)} \ \ \text{and} \ \ \mathbf{r}(l) = \mathbf{X} \ \ \text{(say)}. \tag{49}$$

An increment of distance $\mathrm{d}\mathbf{r}$ *along the path* can be expressed as an ordered pair of *physical* components (i.e. coefficients of local orthogonal unit vectors) which we write as

$$\mathrm{d}\mathbf{r} = (ac(\tau)\mathrm{d}q, a\mathrm{d}\tau). \tag{50}$$

where $q = \lambda$, $\tau = \phi$ and the function $c(\tau) = \cos\tau$. The local unit tangent to a piecewise smooth path described by functions $q(s)$ and $\tau(s)$ is the vector

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = a\left(c\frac{\mathrm{d}q}{\mathrm{d}s}, \frac{\mathrm{d}\tau}{\mathrm{d}s}\right). \tag{51}$$

Again the components on the right are the coefficients of local orthogonal unit vectors.

Equation (33) suggests that we next construct the integral of the Coriolis parameter along the finite segment of the path $\mathbf{r} = \mathbf{r}(s)$ defined above, between the end points (49). This is

$$A = \int_0^l f[\mathbf{r}(s)]\mathrm{d}s \tag{52}$$

between $\mathbf{x}$ and $\mathbf{X}$. We shall see in the next section that A has some features in common with the *action integral* found in classical mechanics. From (51) we deduce, from our hypotheses that the local unit vectors are orthogonal, that

$$\mathrm{d}s^2 = a^2(c^2\mathrm{d}q^2 + \mathrm{d}\tau^2) \tag{53}$$

so that A can be rewritten symbolically as

$$A = a \int_{\mathbf{x}}^{\mathbf{X}} f(c^2 \mathrm{d}q^2 + \mathrm{d}\tau^2)^{\frac{1}{2}}. \tag{54}$$

This symbolic form highlights the presence of the end points (49) in a different way to (52). We will write L for the integrand of A/a in (54) for use in section 4. It is clear that L is bounded at each point on the path. In the special case of constant f,

$$A = lf. \tag{55}$$

The value of A in (52) clearly depends on the path chosen. We *define* $d(\mathbf{x}, \mathbf{X})$ to be the minimum value of $A(\mathbf{x}, \mathbf{X})$ over geometrically possible paths joining $\mathbf{x}$ to $\mathbf{X}$. These paths are geodesics on the sphere rescaled by the Coriolis parameter, and we will show in the next subsection that they are the same as the steepest descent paths (45). We will show that the condition for

$$M = \frac{1}{2} \int_D \left(d(\mathbf{x}, \mathbf{X})^2 + gh \right) h \mathrm{d}\Sigma \tag{56}$$

to be minimised, under variations which have the properties (10) and $\delta\sigma = 0$, where $\sigma = h \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$, defines $\mathbf{x}$ implicitly as a function of $\mathbf{X}$. D denotes either the whole surface of the sphere or a bounded region of it; as in section 2.2. We will show that the equation for the solution value of $\mathbf{X}$ in terms of $\mathbf{x}$ can be interpreted as a state of geostrophic balance, with energy equal to the minimising value of M. We will thus have achieved a generalisation of the geostrophic coordinate transformation.

3.3 Existence of an energy minimiser for the spherical case

In this subsection we assume the existence of a transformed coordinate system (Λ, Φ) as described in section 3.1, but that its relation to the physical coordinates (λ, ϕ) is unknown. Following (44), we define σ to be

$$\sigma = h \frac{\partial(\lambda, \phi)}{\partial(\Lambda, \Phi)} \frac{\cos\phi}{\cos\Phi} \tag{57}$$

and assume that σ is a given function of (Λ, Φ). We then seek to use this information to find (λ, ϕ) as a function of (Λ, Φ). As in sections 2.5 and 3.1, we represent this function as a mapping $\mathbf{s}$ from the surface of the sphere to itself which takes a point $\mathbf{X}$ with spherical coordinates equal to (Λ, Φ) to a point $\mathbf{x}$ with coordinates (λ, ϕ). For this to be useful, we will need this mapping to be invertible, so that we can write $\mathbf{X} = \mathbf{s}^{-1}(\mathbf{x})$. The information given is not sufficient to do this uniquely so, as for the f-plane case in section 2.5, we show that a unique invertible solution can be obtained by minimising the energy (56) subject to variations satisfying (10) and $\delta\sigma = 0$. We can write (56) as an integral over the transformed coordinates as

$$M = \tfrac{1}{2} \int_D (d(\mathbf{s}(\mathbf{X}), \mathbf{X})^2 + gh(\mathbf{s}(\mathbf{X})))\sigma(\mathbf{X}) \mathrm{d}\nu, \tag{58}$$

where $\mathrm{d}\nu$ is the area measure

$$\mathrm{d}\nu = a^2 \cos\Phi \mathrm{d}\Lambda \mathrm{d}\Phi. \tag{59}$$

The discussion in section 3.1 shows that variations satisfying $\delta\sigma = 0$ will only be equivalent to variations satisfying (11) for displacements along steepest descent paths in energy.

Since we cannot use (11), it is not immediately clear that our procedure will yield a solution of the original equations (3), (5), (6). This question is dealt with in section 4.

The constraint $\delta\sigma = 0$ is difficult to enforce. It can be made mathematically tractable using the concept of measure-preserving mappings. (See, for instance, Douglas (2002) for formal definitions). Given a (Borel) set $B \subset \mathcal{S}^2$, define the measures

$$\begin{aligned} \nu(B) &= \int_B \mathrm{d}\nu = \int_B a^2 \cos\Phi \mathrm{d}\Lambda \mathrm{d}\Phi \\ \varpi(B) &= \int_B a^2 h \cos\phi \mathrm{d}\lambda \mathrm{d}\phi \end{aligned} \tag{60}$$

Thus $\varpi(B)$ measures the physical mass of fluid contained in B and $\nu(B)$ measures the area of B in the transformed coordinates. Calculate the image of each point on the sphere by setting $\mathbf{X} = \mathbf{s}^{-1}(\mathbf{x})$. Then, given a (Borel) set B, we can calculate $\varpi(\mathbf{x} : \mathbf{s}^{-1}(\mathbf{x}) \in B)$. The constraint on σ becomes the statement that

$$\varpi(\mathbf{x} : \mathbf{s}^{-1}(\mathbf{x}) \in B) = \int_B \sigma(\Lambda, \Phi) \mathrm{d}\nu \tag{61}$$

Any map $\mathbf{s}$ that satisfies this condition for all (Borel) sets B is called a measure-preserving mapping from $\mathcal{S}^2$ (endowed with measure $\sigma\nu$) to $\mathcal{S}^2$ (endowed with measure ϖ). We will use the concept of the set S of all such mappings to describe our class of variations. We make the assumption (of non-degeneracy) that σ is ν-integrable, which means in particular that $\varpi(\mathbf{x} : \mathbf{s}^{-1}(\mathbf{x}) \in B) > 0$ implies $\nu(B) > 0$ for all sets B.

Minimising (58) over the set of measure-preserving mappings S is an example of a general class of problems called *optimal mass transfer problems*; one seeks an optimal measure-preserving strategy which minimises the "transportation cost", where optimality is measured against a cost function. A review of these problems is given by Gangbo and McCann (1996). The integrand in (58) is an example of a cost function. This problem has a long history and has found many applications in physics, economics and statistics; the original problem posed by Gaspard Monge in 1781 was how to transport material between two locations in the most efficient way. McCann (2001) proves that this problem can be solved uniquely when the integrand in the cost function takes the form of the square of a distance function on a riemannian manifold. The form of (58) suggests that the first term of the integrand could be interpreted in this way. However, we will need to extend McCann's results to deal with the whole of (58) and the fact that our constraint set is not a single set S of measure-preserving mappings.

We mainly discuss the case where D is the whole spherical surface $\mathcal{S}^2$. The modifications to the argument where D is a bounded subset of it are discussed after the proof of Theorem 5. The next step is therefore to define a manifold $\mathcal{M}$ to be the surface of the sphere $\mathcal{S}^2$ endowed with the distance function $d(\mathbf{X}, \mathbf{x})$ defined in the previous subsection. We will assume that d is a twice continuously differentiable function of $\mathbf{X}$ and $\mathbf{x}$. To apply McCann's theorem, we need $\mathcal{M}$ to be a compact, connected manifold without boundary. Since the Coriolis parameter f defines the distance function in (52), and f goes to zero at the equator, these assumptions will not be satisfied. We therefore regularise the problem, so that the assumptions are satisfied. Having solved the regularised problem, we will then have to show that a solution of the original problem can be recovered in the limit as the parameter defining the regularisation tends to zero.

Define the riemannian manifold $\mathcal{M}_\epsilon$ to be the surface of the sphere together with the metric whose components, $\hat{g}_{ij}$, take the form

$$\hat{g}_{ij}^{S^2} = \mathcal{F}^2(\phi) g_{ij}^{S^2}, \; \hat{g}^{ij S^2} = \mathcal{F}^{-2}(\phi) g^{ij S^2}, \tag{62}$$

where $g_{ij}^{S^2}$ denotes the usual components of the metric on a sphere S^2 of radius a

$$g_{ij}^{S^2} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \cos^2 \phi \end{pmatrix}. \tag{63}$$

$\mathcal{F}(\phi)$ is chosen to be a smooth modification of the function $2|\Omega \sin \phi|$ which is a twice differentiable function of ϕ, is equal to $2|\Omega \sin \phi|$ for $\phi > \eta > 0$ for some (small) η and has a minimum value $\epsilon > 0$. The regularisation is equivalent to using $f_\epsilon = \mathcal{F}(\phi(\mathbf{r}))$ in (52) to calculate A, and defining the distance function d by finding the minimum value of A. At a later stage, when the complete solution has been obtained, we will consider the effect of letting $\eta, \epsilon \to 0$. The resulting manifold $\mathcal{M}_\epsilon$ is topologically equivalent to the sphere.

We note in passing that the general setting for a treatment of geometries defined by metrics of the form given in (62) is known as *conformal geometry* because multiplying metrics of the type $g_{ij}^{S^2}$ by the functions $\mathcal{F}(\phi)$ preserves angles and ratios of distances in the new geometries defined by $\hat{g}_{ij}^{S^2}$ etc.; see Sewell (2002, §2) for further comment on the subject of conformal transformations.

Noting that we have assumed that σ is ν-integrable, McCann (2001, Theorem 8) yields that

$$C(\mathbf{s}) = \tfrac{1}{2} \int_{\mathcal{M}_\epsilon} d^2(\mathbf{s}(\mathbf{X}), \mathbf{X}) \sigma(\mathbf{X}) \mathrm{d}\nu. \tag{64}$$

has a unique minimiser $\mathbf{t}$ over $\mathbf{s} \in S$, where S is as defined following (61), which can be expressed in the form

$$\mathbf{t}(\mathbf{X}) = \exp_{\mathbf{X}}[-\nabla \Psi(\mathbf{X})], \tag{65}$$

where Ψ is a scalar function and the gradient is taken with respect to the metric on $\mathcal{M}_\epsilon$. The operator on the right hand side of (65) is the *exponential map* (see, for example, Schutz 1980 §2.13). The 'exp' notation is a shorthand for the differential operator, which when applied to an analytic function gives the Taylor series of that function expanded about some datum point. An explicit example is provided by semi-geostrophic theory on an f-plane. The dual coordinates (35) can be expressed in the form

$$X_i = \exp\left(\epsilon \frac{\mathrm{d}}{\mathrm{d}s}\right) x_i, \tag{66}$$

where $X_i = (X, Y)$, $x_i = (x, y)$, s parametrizes the path between the two end points $(\mathbf{X}, \mathbf{x})$, and ϵ is a distance along the path. For the f-plane case the path is a straight line, with metric given by (33), and a simple relationship can be derived relating the normal to the path, ∇d, to the tangent vector $\mathrm{d}x_i/\mathrm{d}s$:

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = -\varepsilon_{ij} \frac{\partial d}{\partial x_j}, \tag{67}$$

where ε_{ij} is the alternating symbol in two dimensions. The derivative $\mathrm{d}/\mathrm{d}s$ defines a vector field $\mathrm{d}/\mathrm{d}s = (\mathrm{d}x_i/\mathrm{d}s)\partial/\partial x_i$, and therefore (66) becomes

$$X_i = \left(1 + \epsilon \frac{\mathrm{d}x_j}{\mathrm{d}s} \frac{\partial}{\partial x_j} + \frac{\epsilon^2}{2!} \left(\frac{\mathrm{d}x_j}{\mathrm{d}s} \frac{\partial}{\partial x_j}\right) \left(\frac{\mathrm{d}x_k}{\mathrm{d}s} \frac{\partial}{\partial x_k}\right) + \ldots\right) x_i. \tag{68}$$

Because the path is a straight line the tangent vector $\mathrm{d}x_i/\mathrm{d}s$ is constant along the path and therefore (68) becomes

$$X_i = x_i + \epsilon\frac{\mathrm{d}x_i}{\mathrm{d}s}. \tag{69}$$

Then, using (67) and (33), (69) becomes

$$X_i = x_i + \frac{\epsilon g}{fd}\frac{\partial h}{\partial x_i} \tag{70}$$

and if $\epsilon = d/f$, we recover (35).

The existence of the function Ψ characterizes optimality of the mapping $\mathbf{t}$ by specifying the direction, given by $\nabla\Psi$, and the distance, given by $|\nabla\Psi|$, in which to move material from $\mathbf{X}$ to other locations in $\mathcal{M}_\epsilon$. The paths over which one moves material are the geodesics between points as defined by the metric on $\mathcal{M}_\epsilon$. Ψ is determined up to an additive constant.

In order to apply McCann's result to our problem, we need to use a number of the arguments he uses to prove the theorem. We summarise these below, referring to his paper for further detail. He shows that the scalar function Ψ can be characterised in the following way. First introduce a conjugate or dual function Ψ^c to Ψ, defined by

$$\Psi^c(\mathbf{x}) = \inf_{\mathbf{X}}(\tfrac{1}{2}d^2(\mathbf{X},\mathbf{x}) - \Psi(\mathbf{X})). \tag{71}$$

Then, in particular, we have

$$\Psi(\mathbf{X}) + \Psi^c(\mathbf{x}) \le \tfrac{1}{2}d^2(\mathbf{X},\mathbf{x}). \tag{72}$$

for any $\mathbf{x}$, $\mathbf{X}$. Ψ^c can be constructed for any choice of Ψ. Next, suppose that the infimum is attained at $\mathbf{X} = \mathbf{w}(\mathbf{x})$. Then

$$\Psi(\mathbf{w}(\mathbf{x})) + \Psi^c(\mathbf{x}) = \tfrac{1}{2}d^2(\mathbf{w}(\mathbf{x}),\mathbf{x}). \tag{73}$$

Now consider the behaviour of $\frac{1}{2}d^2(\mathbf{X},\mathbf{x}) - \Psi^c(\mathbf{x})$ as a function of $\mathbf{x}$. Equation (72) applied at $\mathbf{X} = \mathbf{w}(\mathbf{x})$ shows that

$$\Psi(\mathbf{w}(\mathbf{x})) \le \tfrac{1}{2}d^2(\mathbf{w}(\mathbf{x}),\mathbf{y}) - \Psi^c(\mathbf{y}). \tag{74}$$

for any $\mathbf{y}$, with equality at $\mathbf{y} = \mathbf{x}$. Thus, assuming any $\mathbf{X}$ can be written as $\mathbf{w}(\mathbf{x})$ for some $\mathbf{x}$, we have that

$$\Psi(\mathbf{X}) = \inf_{\mathbf{y}}(\tfrac{1}{2}d^2(\mathbf{X},\mathbf{y}) - \Psi^c(\mathbf{y})). \tag{75}$$

Since this is just the definition of $(\Psi^c)^c$, we can write $(\Psi^c)^c = \Psi^{cc} = \Psi$. This is an example of Theorem 1 of Sewell (2002, p.147): if we define a transformation between $\mathbf{x}$ and $\mathbf{X}$ using (73), then if the transformation is invertible, we will have $\Psi^{cc} = \Psi$. However, the converse is only true under a non-degeneracy assumption that $\mathbf{w}$ does not map sets of positive size to zero size. Our condition that σ is ν-integrable is equivalent to this. For convenience we henceforth call such a function *involutive* since the property $\Psi^{cc} = \Psi$ is an involutive property. For a given distance function $\frac{1}{2}d^2$, this property is only satisfied for Ψ which satisfy particular geometrical properties. McCann (2001) calls such a function $\frac{1}{2}d^2$-*concave*. This definition is discussed further (for a more general cost function) in Rachev and Ruschendorf (1998), section 3.3; McCann has additional results for this particular case.

McCann proves (in his Theorem 7) that, if $\mathbf{t}$ is the minimising map (65) with scalar function Ψ, then at every $\mathbf{x}$ where Ψ is differentiable the inequality in (72) is strict unless $\mathbf{x} = \mathbf{t}(\mathbf{X})$, when it holds with equality. Thus

$$\Psi(\mathbf{X}) + \Psi^c(\mathbf{t}(\mathbf{X})) = \tfrac{1}{2}d^2(\mathbf{X}, \mathbf{t}(\mathbf{X})). \tag{76}$$

He proves that Ψ is involutive and thence, because of the non-degeneracy condition, that $\mathbf{t}$ is invertible almost everywhere. It follows that $\mathbf{t}^{-1}(\mathbf{x}) = \exp_{\mathbf{X}}(-\nabla\Psi^c(\mathbf{x}))$. To link with Theorem 2, we make the identification $\Psi_c(\mathbf{x}) = -gh(\mathbf{x})$.

In euclidean space, where d is euclidean distance, the involutivity requirement will be satisfied if $\frac{1}{2}d^2(\mathbf{X}, \mathbf{x}) - \Psi(\mathbf{X})$ is a strictly convex function of $\mathbf{X}$ for each $\mathbf{x}$, ensuring that $\mathbf{t}(\mathbf{x})$ is single-valued. Because of the simple analytic form of $\frac{1}{2}d^2$, it is easy to show that this condition will be satisfied if $\frac{1}{2}d^2(\mathbf{X}, 0) - \Psi(\mathbf{X})$ is a strictly convex function. In our case, where we are working on a manifold topologically equivalent to the sphere, the concept of a globally convex function does not make sense. However, a convex function in euclidean space may be characterised as the supremum of a family of continuous affine functionals: the corresponding condition on Ψ is that it is equal to the infimum (over $\mathbf{x}$) of a family of functions of the form $\frac{1}{2}d^2(\mathbf{X}, \mathbf{x}) + \phi(\mathbf{x})$ for some function ϕ, as in (75). The latter property does still make sense on a manifold topologically equivalent to the sphere. This will be the geometric condition that $\mathbf{t}$ is invertible. Using this, McCann proves that involutive functions satisfy analogous regularity properties to convex functions, which is crucial in the rest of our arguments.

We now prove that the mapping $\mathbf{t}$ minimises the notional energy (58) which includes the potential energy term, using an argument similar to that in Cullen and Gangbo (2001) for the f-plane case. The strategy is as follows: we examine the energy functional when evaluated with the minimizer of (64) and show that perturbations always generate positive increments to this functional.

We first make some additional definitions. Write μ for the area measure in physical space, so that

$$\mu(B) = \int_B a^2 \cos\phi \mathrm{d}\lambda \mathrm{d}\phi \tag{77}$$

and, for a $\mu-$integrable function $\eta : \mathcal{M}_\epsilon \to I\!\!R$, write $\eta\mu$ for the measure defined by

$$\eta\mu(B) = \int_B a^2 \eta(\lambda, \phi) \cos\phi \mathrm{d}\lambda \mathrm{d}\phi \tag{78}$$

for (Borel) sets $B \subset \mathcal{M}_\epsilon$. Thus, the measure ϖ can be written as $h\mu$. Similarly define $\sigma\nu$ for $\nu-$integrable $\sigma : \mathcal{M}_\epsilon \to I\!\!R$. We assume that a $\nu-$integrable σ is given.

We call a pair $(\eta, \mathbf{s})$ *admissible* if $\mathbf{s}$ is an invertible measure-preserving mapping from $(\mathcal{M}_\epsilon, \sigma\nu)$ to $(\mathcal{M}_\epsilon, \eta\mu)$. We think of $\mathbf{s}^{-1}$ as a (possible) coordinate transformation, and η as a (possible) height function. We write $\mathcal{S} \times \mathcal{H}$ for the set of admissible pairs. We will show that there is a unique pair $(\mathbf{t}, h) \in \mathcal{S} \times \mathcal{H}$ which minimises

$$M(\mathbf{s}, \eta) = \tfrac{1}{2} \int_{\mathcal{M}_\epsilon} (d(\mathbf{s}(\mathbf{X}), \mathbf{X})^2 + g\eta(\mathbf{s}(\mathbf{X})))\sigma(\mathbf{X})\mathrm{d}\nu. \tag{79}$$

over $(\mathbf{s}, \eta) \in \mathcal{S} \times \mathcal{H}$. This is equivalent to minimising (58) integrated over $\mathcal{M}_\epsilon$ under the constraints (10) and $\delta\sigma = 0$. The desired coordinate transformation will be defined by the mapping $\mathbf{t}^{-1}$.

The problem of minimising M cannot be rewritten as a standard mass transfer problem as solved by McCann (2001). However, his theorem can be used in the proof. We begin by fixing η, and considering only the first part of the integrand in (79). This allows us to determine a $\mathbf{t}_\eta$ that depends on η. We then show that (79) can be minimised as a function of η. If h is the choice of η that achieves the minimisation, then $(\mathbf{t}_h, h)$ solves the full problem.

Theorem 4 *The integral (79) is uniquely minimised for $(\mathbf{s}, \eta) \in \mathcal{S} \times \mathcal{H}$ by $(\mathbf{t}, h)$, where $\mathbf{t}$ is the map (65) that minimises (64) and $-gh = \Psi^c$. Ψ^c is defined by (71) using the Ψ that appears in (65).*

Proof Start with any $\eta(\mathbf{x}) \geq 0$ such that $\int_{\mathcal{M}_\epsilon} \eta \mathrm{d}\mu$ is fixed. Use McCann's theorem to construct a map $\mathbf{t}_\eta$ which minimises $C(\mathbf{s})$ for $\mathbf{s} : (\mathcal{M}_\epsilon \sigma\nu) \to (\mathcal{M}_\epsilon \eta\mu)$ as in (64). By construction, $(\mathbf{t}_\eta, \eta)$ is then an admissible pair. Use (79) to calculate $M(\mathbf{t}_\eta, \eta)$. Following arguments of Cullen and Gangbo (2001), this will be a strictly convex function of η, (the second term is clearly strictly convex), and so can be uniquely minimised by some choice of η.

For any η, we can find $\mathbf{t}_\eta$ and hence Ψ^c using (71) and (65). We now make the 'guess' that the minimiser is characterised by setting $-g\eta = \Psi^c$, which is consistent with the analysis in Theorem 1. For this choice, write $h = \eta$ and $\mathbf{t} = \mathbf{t}_h$. We demonstrate that this characterisation indeed gives a minimiser as follows. Let $(\mathbf{s}, \eta)$ be an arbitrary member of $\mathcal{S} \times \mathcal{H}$. The definition (71) with $\mathbf{x}$ chosen to be $\mathbf{s}(\mathbf{X})$ gives

$$\Psi(\mathbf{X}) + \Psi^c(\mathbf{s}(\mathbf{X})) \leq \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{s}(\mathbf{X})). \tag{80}$$

Now integrate (80) with respect to the measure $\sigma\nu$ to give

$$\int_{\mathcal{M}_\epsilon} \Psi(\mathbf{X}) \sigma \mathrm{d}\nu + \int_{\mathcal{M}_\epsilon} \Psi^c(\mathbf{s}(\mathbf{X})) \sigma \mathrm{d}\nu \leq \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{s}(\mathbf{X})) \sigma \mathrm{d}\nu \tag{81}$$

The inequality is strict if $\mathbf{s} \neq \mathbf{t}$. Now using the fact that $\mathbf{s} : (\mathcal{M}_\epsilon, \sigma\nu) \to (\mathcal{M}_\epsilon, \eta\mu)$ is measure-preserving, and identifying Ψ^c with $-gh$, we have that

$$\int_{\mathcal{M}_\epsilon} \Psi(\mathbf{X}) \sigma \mathrm{d}\nu - \int_{\mathcal{M}_\epsilon} gh\eta \mathrm{d}\mu \leq \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{s}(\mathbf{X})) \sigma \mathrm{d}\nu \tag{82}$$

with strict inequality if $\mathbf{s} \neq \mathbf{t}$. A similar calculation replacing $\mathbf{s}$ with $\mathbf{t}$ and η with h gives

$$\int_{\mathcal{M}_\epsilon} \Psi(\mathbf{X}) \sigma \mathrm{d}\nu - \int_{\mathcal{M}_\epsilon} gh^2 \mathrm{d}\mu = \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{t}(\mathbf{X})) \sigma \mathrm{d}\nu. \tag{83}$$

Now

$$\begin{aligned} M(\mathbf{s}, \eta) - M(\mathbf{t}, h) = & \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{s}(\mathbf{X})) \sigma \mathrm{d}\nu - \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{t}(\mathbf{X})) \sigma \mathrm{d}\nu + \\ & \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} g\eta(\mathbf{s}(\mathbf{X})) \sigma \mathrm{d}\nu - \int_{\mathcal{M}_\epsilon} \tfrac{1}{2} gh(\mathbf{t}(\mathbf{X})) \sigma \mathrm{d}\nu \end{aligned} \tag{84}$$

The first two integrals in (84) are estimated using (82) and (83) respectively. In the second two integrals we replace $\sigma \mathrm{d}\nu$ by $\eta \mathrm{d}\mu$ and $h\mathrm{d}\mu$ respectively (noting $\mathbf{s}$ and $\mathbf{t}$ are measure preserving). This gives

$$\begin{aligned} M(\mathbf{s},\eta) - M(\mathbf{t},h) &\geq g\int_{\mathcal{M}_\epsilon}(h^2-\eta h)\mathrm{d}\mu + \int_{\mathcal{M}_\epsilon}\tfrac{1}{2}g\eta^2\mathrm{d}\mu - \int_{\mathcal{M}_\epsilon}\tfrac{1}{2}gh^2\mathrm{d}\mu \\ &= \tfrac{1}{2}g\int_{\mathcal{M}_\epsilon}(h-\eta)^2\mathrm{d}\mu \end{aligned} \tag{85}$$

Thus $M(\mathbf{s},\eta) - M(\mathbf{t},h) > 0$ unless $(\mathbf{s},\eta)$ is equal to $(\mathbf{t},h)$. The result follows. □

We can then deduce

Theorem 5 *The integral (56), with f_ϵ as defined following (63), and d defined by the minimum of A in (52), is minimised with respect to displacements satisfying (10) and $\delta\sigma = 0$ if $\mathbf{X}$ is given as a function of $\mathbf{x}$ by the map*

$$\mathbf{X} = \exp_{\mathbf{x}}[g\nabla h(\mathbf{x})] \tag{86}$$

where $-gh(\mathbf{x})$ is an involutive function. The minimising value is

$$\frac{1}{2}\int_{\mathcal{S}^2}\left(\mathbf{u}_g^{*2} + gh\right)h\mathrm{d}\mu \tag{87}$$

where $\mathcal{F}\mathbf{u}_g^* = (\frac{1}{a\cos\phi}\frac{\partial h}{\partial\lambda}, \frac{1}{a}\frac{\partial h}{\partial\phi})$. In particular $\mathbf{u}_g^* = \mathbf{u}_g$ on those parts of $\mathcal{M}_\epsilon$ where $\mathcal{F} = f$.

Proof. The definition of the space of admissible pairs $\mathcal{S}\times\mathcal{H}$ is consistent with $\delta\sigma = 0$. The definitions of the measures $\eta\mu$ and $h\mu$ are consistent with (10). The integral (56) takes the values $M(\mathbf{s},\eta)$ given the map $\mathbf{s}(\mathbf{X})$, and the value $M(\mathbf{t},h)$ given the map $\mathbf{t}(\mathbf{X})$. Theorem (4) shows that $M(\mathbf{s},\eta) - M(\mathbf{t},h) > 0$ if $(\mathbf{s},\eta) \neq (\mathbf{t},h)$ and that the inverse map $\mathbf{t}^{-1}$ takes the form (86), as required. Since McCann's theorem proves that Ψ is involutive, so is $-gh$. Furthermore, the statement $\mathbf{t}^{-1}(\mathbf{x}) = \exp_{\mathbf{x}}[\nabla gh(\mathbf{x})]$ implies that the magnitude of $\mathbf{t}^{-1}(\mathbf{x})$ is a 'distance' $\mathcal{F}^{-1}\nabla(gh(\mathbf{x}))$ along the geodesic starting from $\mathbf{x}$. To see this, we note that $|g\nabla h| = (g^2\hat{g}^{ijS^2}\nabla_i h\nabla_j h)^{\frac{1}{2}} = (g^2\mathcal{F}^{-2}\nabla^i h\nabla_i h)^{\frac{1}{2}}$ (using (62)). The minimising value of $\frac{1}{2}d^2(\mathbf{t}^{-1}(\mathbf{x}),\mathbf{x})$ is $\frac{1}{2}|g\nabla h(\mathbf{x})|^2 = \frac{1}{2}g^2\mathcal{F}^{-2}\nabla^i h\nabla_i h = \frac{1}{2}\mathbf{u}_g^{*2}$ as we require. □

This theorem proves the claim made at the end of section 2.4, subject to the regularisation of the problem made by replacing f with $\mathcal{F}$. If the physical domain is a subset D of the surface of the sphere, as would arise in oceanographic applications, we can make the same definitions as above, including the use of the regularised Coriolis parameter $\mathcal{F}$. The support of σ will be some subset of $\mathcal{S}^2$, and we seek a mapping from $\mathcal{S}^2$ into D which minimises (56) where the integral is taken over D. If the domain is small enough and far enough removed from the equator (so that all points are geodesically linked), the regularisation will not be needed. In that case, Theorem 13 of McCann (2001) can be used to show that an optimal map can be found.

The characterisation of $-gh$ as involutive is equivalent to the local inertial stability condition on the matrix (24) derived earlier, as can be seen by substituting for u, v in terms of h using the geostrophic relation, symmetrizing the matrix, and calculating the determinant. We obtain a term f^2 which represents the local convexity of the distance function $d(\mathbf{x},\mathbf{y})$ for a fixed $\mathbf{y}$, several terms containing f and its spatial derivative along with first derivatives of h, and terms with second derivatives of h. The second derivative terms measure the convexity of h, giving the condition that $gh(\mathbf{x}) + d(\mathbf{x}, \mathbf{y})$ is locally strictly convex as a function of $\mathbf{x}$ for each $\mathbf{y}$, which is a necessary condition for it to have a unique minimiser for every choice of $\mathbf{y}$. Locally, $d(\mathbf{x}, \mathbf{y})$ is well approximated by $\frac{1}{2}f^2|\mathbf{x}-\mathbf{y}|^2$, and the involutive property of $-gh$ at y is achieved as a result of the strict convexity of $\frac{1}{2}f^2|\mathbf{x}-\mathbf{y}|^2 + gh$, which is the convexity principle used in the f-plane case.

The procedure described in section 3.1 seeks to minimise the energy subject to the conditions stated in Theorem 5. Since Theorem 5 guarantees a unique minimiser, any solution found by the procedure must be the correct one. Consider a fixed $\mathbf{X}$ and a displacement $\delta\mathbf{r}$ of $\mathbf{x}$ along the geodesic path connecting $\mathbf{x}$ to $\mathbf{X}$. Since the geodesic is normal to $\mathbf{u}_g$ at $\mathbf{x}$, and the geodesic distance from $\mathbf{x}$ to $\mathbf{X}$ is $|\mathbf{u}_g|$, we have

$$f\delta\mathbf{r} = (-\delta v_g, \delta u_g) \tag{88}$$

This is exactly (11), showing that, for displacements along the geodesic, (11) is equivalent to $\delta\mathbf{X} = 0$. Thus the geodesic must coincide with the energy minimisation path defined in section 3.1, equation (45).

4 Solution of the evolution equations using the new coordinate transformation

4.1 Differential properties of the distance function on the sphere

We now need to establish certain differential properties of the path integral (54) that appears in the kinetic energy (56). Given a path $\mathbf{r} = \mathbf{r}(s)$, of length l, between end points $\mathbf{x}$ and $\mathbf{X}$ in (49), and the associated value of A, we wish to allow the path, and in particular its end points, to vary, and to examine the effect on A.

First, if only the length l varies, along the local tangent to the end point $\mathbf{X}$, with the starting point $\mathbf{x}$ held fixed, then

$$\frac{\mathrm{d}A}{\mathrm{d}l} = f(\mathbf{X}) \tag{89}$$

immediately from (52) (and (55) provides special examples). Since

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}l} \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} \tag{90}$$

from (49) is the unit tangent (and a particular value of (51)) at the end, we can construct a *vector* gradient

$$\frac{\mathrm{d}A}{\mathrm{d}\mathbf{X}} \equiv \frac{\mathrm{d}A}{\mathrm{d}l}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}l} = f(\mathbf{X})\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}l} \tag{91}$$

of A at the end, for this particular *variation*, of l alone.

More generally, we now imagine that the *direction* of the local tangent at the end $\mathbf{X}$ is allowed to vary, as well as the length l of the curve. The curve becomes piecewise smooth there (instead of smooth as just above), but the same construction of the vector gradient (91) can be repeated, but using the *new* end tangent vector.

A different proof of this last conclusion can be constructed using hamiltonians as follows. The vector gradient is first defined by specifying its components with respect to local orthogonal unit vectors as

$$\frac{\mathrm{d}A}{\mathrm{d}\mathbf{X}} \equiv \frac{1}{a}\left(\frac{1}{c}\frac{\partial A}{\partial q}, \frac{\partial A}{\partial \tau}\right) \tag{92}$$

where, on the right, the partial derivatives are with respect to the *end* values of q and τ. *Both* of q and τ will be available almost everywhere on the curve to act as the local path parameter, as an alternative to s, and we shall write $\tau' = \mathrm{d}\tau/\mathrm{d}q$ and $\dot{q} = \mathrm{d}q/\mathrm{d}\tau$ for brevity, so that $\tau'\dot{q} = 1$. The exceptional points will be where the path is locally parallel to a q-coordinate curve so that only q is available there (thus avoiding infinite $\dot{q}$); and where the path is locally parallel to a τ-coordinate curve so that only τ is available there (thus avoiding infinite τ'). The modified Coriolis parameter that appears in (62) is the function $\mathcal{F}(\tau)$ with $\tau = \phi$. The integrand of A/a can therefore be written as either of the functions

$$\mathsf{L}(\tau', \tau) = \mathcal{F}(\tau)(c^2 + \tau'^2)^{\frac{1}{2}} \quad \text{or} \quad L(\dot{q}, \tau) = \mathcal{F}(\tau)(1 + c^2\dot{q}^2)^{\frac{1}{2}} \tag{93}$$

when q or τ is the path parameter, respectively. At each τ, the functions $\mathsf{L}(\tau')$ and $L(\dot{q})$ are strictly convex, being one branch of a hyperbola in each case.

Either of (93) can be used as a lagrangian to define a momentum $\mathsf{p} = \partial\mathsf{L}/\partial\tau'$ or $p = \partial L/\partial\dot{q}$ and, *via* the standard Legendre transformation, a hamiltonian $\mathsf{H}(\mathsf{p}, \tau) = \mathsf{p}\tau' - \mathsf{L}$ or $H(p, \tau) = p\dot{q} - L$, such that $\tau' = \partial\mathsf{H}/\partial\mathsf{p}$ and $\dot{q} = \partial H/\partial p$. Since $\mathcal{F} \neq 0$ and away from the poles, so that $c > 0$ we find that

$$\mathsf{H}(\mathsf{p}, \tau) = -c(\mathcal{F}^2 - \mathsf{p}^2)^{\frac{1}{2}} \quad \text{and} \quad H(p, \tau) = -\frac{(\mathcal{F}^2c^2 - p^2)^{\frac{1}{2}}}{c} \tag{94}$$

The positive square root is chosen throughout these calculations. Then at each τ, the functions $\mathsf{H}(\mathsf{p})$ and $H(p)$ are strictly convex.

We can now write (54) as

$$\frac{A}{a} = -\int \mathsf{H}\mathrm{d}q + \int \mathsf{p}\mathrm{d}\tau \quad \text{or} \quad \frac{A}{a} = \int p\mathrm{d}q - \int H\mathrm{d}\tau \tag{95}$$

between limits of integration which are those values of q and τ corresponding to the end points $\mathbf{x}$ and $\mathbf{X}$, i.e. to $s = 0$ and $s = L$. Differentiating with respect to those end values in (92) gives, at $\mathbf{X}$,

$$\frac{\partial A}{\partial \mathbf{X}} = \left(-\frac{\mathsf{H}}{c}, \mathsf{p}\right) \quad \text{or} \quad \left(\frac{p}{c}, -H\right). \tag{96}$$

We can write (51) at $\mathbf{X}$ as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} = \frac{\mathcal{F}}{\mathsf{L}}(c, \tau') \quad \text{or} \quad \frac{\mathcal{F}}{L}(c\dot{q}, 1) \tag{97}$$

because $\mathcal{F}\mathrm{d}s = a\mathsf{L}\mathrm{d}q$ or $aL\mathrm{d}\tau$ respectively, i.e. $\mathrm{d}q/\mathrm{d}s = \mathcal{F}/a\mathsf{L}$ or $\mathrm{d}\tau/\mathrm{d}s = \mathcal{F}/aL$ when q or τ is taken as the independent variable, respectively.

Equating components, we see that the first of (91) holds if and only if

$$-\frac{\mathsf{H}}{c} = \frac{\mathcal{F}^2c}{\mathsf{L}} \quad \text{and} \quad \mathsf{p} = \frac{a\mathcal{F}^2\tau'}{\mathsf{L}} \tag{98}$$

in the first case, and if and only if

$$\frac{p}{c} = \frac{\mathcal{F}^2c\dot{q}}{L} \quad \text{and} \quad -H = \frac{\mathcal{F}^2}{L} \tag{99}$$

in the second case.

It is easy to verify that (98) and (99) are satisfied by using the properties $\mathsf{p} = \partial \mathsf{L}/\partial \tau'$, $\mathsf{H} = \mathsf{p}\tau' - \mathsf{L}$, $p = \partial H/\partial \dot{q}$ and $H = p\dot{q} - L$ with (93).

We can show, using similar techniques, that a similar result holds for $\partial A/\partial \mathbf{x}$:

$$\frac{\partial A}{\partial \mathbf{x}} = \left(\frac{\mathsf{H}}{c}, -\mathsf{p}\right) \quad \text{or} \quad \left(-\frac{p}{c}, H\right). \tag{100}$$

4.2 The semi-geostrophic equations in transformed coordinates on the sphere

From (73) *et seq.* and Theorem 4, we identify $\Psi^c(\mathbf{x}, t) = -gh(\mathbf{x}, t)$ and define $\Psi(w(\mathbf{x}), t) \equiv gH(\mathbf{X}, t)$ so that

$$gH(\mathbf{X}, t) - gh(\mathbf{x}, t) = \tfrac{1}{2} d^2(\mathbf{X}, \mathbf{x}). \tag{101}$$

By differentiating (101) along fluid trajectories in physical space, we shall derive the relationship between the equations of motion (6) in physical space coordinates and the image of those equations in the new transformed variables. Referring to (101), note that $d(\mathbf{X}, \mathbf{x})$ depends on time only through the time dependence of $\mathbf{X}$ and $\mathbf{x}$, we have

$$d\frac{\partial d}{\partial t} = 0 = g\frac{\partial H}{\partial t} - g\frac{\partial h}{\partial t}.$$

This is a consequence of the passive variable nature of t in the duality expressed by (101).

Adopting τ as the path parameter, then using $(96)_2$, $(100)_2$, and $(99)_2$ with $A = d$, we have, from (101),

$$d\frac{\partial d}{\partial \mathbf{r}}\bigg|_{\mathbf{r}=\mathbf{X}} = g\frac{\partial H}{\partial \mathbf{X}} = d\left(\frac{p}{c}, \frac{\mathcal{F}^2}{L}\right)\bigg|_{\mathbf{r}=\mathbf{X}}, \tag{102}$$

$$d\frac{\partial d}{\partial \mathbf{r}}\bigg|_{\mathbf{r}=\mathbf{x}} = -g\frac{\partial h}{\partial \mathbf{x}} = -d\left(\frac{p}{c}, \frac{\mathcal{F}^2}{L}\right)\bigg|_{\mathbf{r}=\mathbf{x}}. \tag{103}$$

Because the lagrangian $L(\tau, \dot{q})$ is independent of q, then, by the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

the momentum $p \equiv \partial L/\partial \dot{q}$ is constant along the path. Hence we can deduce the following relationships between the gradients of h and H: from $(96)_2$ and $(100)_2$ (using the notation $\mathbf{X} = (\Lambda, \Phi), \mathbf{x} = (\lambda, \phi)$)

$$\frac{g}{a\cos\Phi}\frac{\partial H}{\partial \Lambda} = \frac{dp}{\cos\Phi}, \quad \frac{g}{a\cos\phi}\frac{\partial h}{\partial \lambda} = \frac{dp}{\cos\phi}$$

and therefore we have (by cross-multiplication)

$$\frac{\partial H}{\partial \Lambda} = \frac{\partial h}{\partial \lambda}, \tag{104}$$

and from

$$\frac{g}{a}\frac{\partial H}{\partial \Phi} = \frac{d\mathcal{F}(\mathbf{X})^2}{L(\mathbf{X})}$$

and

$$\frac{g}{a}\frac{\partial h}{\partial \phi} = \frac{d\mathcal{F}(\mathbf{x})^2}{L(\mathbf{x})}$$

(where the functional dependence of $\mathcal{F}$ and L on $\mathbf{x}$ and $\mathbf{X}$ means that these functions are evaluated at the respective end points) we have

$$\frac{\mathcal{F}(\mathbf{x})^2}{L(\mathbf{x})}\frac{\partial H}{\partial \Phi} = \frac{\mathcal{F}(\mathbf{X})^2}{L(\mathbf{X})}\frac{\partial h}{\partial \phi}. \tag{105}$$

Note that, when f is a constant and the path is a straight line (which means that $L(\mathbf{x}) = L(\mathbf{X})$)), then in cartesian coordinates (104) and (105) reduce to $\partial H/\partial \mathbf{X} = \partial h/\partial \mathbf{x}$, which is the usual gradient property of the geostrophic momentum transformation.

We now construct an identity involving two components of a vector field and deduce that in order for the identity to hold for an arbitrary geostrophic flow, the semi-geostrophic equations in physical space must be satisfied (subject to the replacement of f by $\mathcal{F}$). This constraint enables us to deduce the form of the equations in the new variables.

Differentiating (101) along fluid trajectories and using Theorem 5, which implies that $d^2/2 = |\mathbf{u}_g^*|^2/2$, gives

$$u_g^*\frac{\mathrm{d}u_g^*}{\mathrm{d}t} + v_g^*\frac{\mathrm{d}v_g^*}{\mathrm{d}t} = g\frac{\partial H}{\partial \mathbf{X}}.\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} - g\frac{\partial h}{\partial \mathbf{x}}.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \tag{106}$$

using the passive variable property of t. Substituting for the definition of geostrophic flow (5) into (106) and writing both sets of dependent variables in terms of spherical polar coordinates, (106) becomes

$$\begin{aligned} -\frac{g}{a\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \phi}\frac{\mathrm{d}u_g^*}{\mathrm{d}t} + \frac{g}{a\cos\phi\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \lambda}\frac{\mathrm{d}v_g^*}{\mathrm{d}t} &= \\ \frac{g}{a\cos\Phi}\frac{\partial H}{\partial \Lambda}U + \frac{g}{a}\frac{\partial H}{\partial \Phi}V - \frac{g}{a\cos\phi}\frac{\partial h}{\partial \lambda}u - \frac{g}{a}\frac{\partial h}{\partial \phi}v&, \end{aligned} \tag{107}$$

where $(U, V) \equiv (a\cos\Phi\mathrm{d}\Lambda/\mathrm{d}t, a\mathrm{d}\Phi/\mathrm{d}t)$ and $(u, v) \equiv (a\cos\phi\mathrm{d}\lambda/\mathrm{d}t, a\mathrm{d}\phi/\mathrm{d}t)$. From (104) and (105) we substitute in (107) for the derivatives of H with respect to Λ and Φ in terms of derivatives of h with respect to λ and ϕ, and group terms with coefficients proportional to $\partial h/\partial \lambda$ and $\partial h/\partial \phi$:

$$\begin{aligned} &-\frac{g}{a\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \phi}\left(\frac{\mathrm{d}u_g^*}{\mathrm{d}t} + \frac{\mathcal{F}(\mathbf{X})^2L(\mathbf{x})}{\mathcal{F}(\mathbf{x})L(\mathbf{X})}V - \mathcal{F}(\mathbf{x})v\right) \\ &+ \frac{g}{a\cos\phi\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \lambda}\left(\frac{\mathrm{d}v_g^*}{\mathrm{d}t} - \mathcal{F}(\mathbf{x})\frac{\cos\phi}{\cos\Phi}U + \mathcal{F}(\mathbf{x})u\right) = 0. \end{aligned} \tag{108}$$

Adding and subtracting a term $\dot{\lambda}u_g^*v_g^*\sin\phi$, (108) becomes

$$\begin{aligned} &-\frac{g}{a\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \phi}\left(\frac{\mathrm{d}u_g^*}{\mathrm{d}t} - \dot{\lambda}v_g^*\sin\phi - \mathcal{F}(\mathbf{x})v + \frac{\mathcal{F}(\mathbf{X})^2L(\mathbf{x})}{\mathcal{F}(\mathbf{x})L(\mathbf{X})}V\right) \\ &+ \frac{g}{a\cos\phi\mathcal{F}(\mathbf{x})}\frac{\partial h}{\partial \lambda}\left(\frac{\mathrm{d}v_g^*}{\mathrm{d}t} + \dot{\lambda}u_g^*\sin\phi - \mathcal{F}(\mathbf{x})u - \mathcal{F}(\mathbf{x})\frac{\cos\phi}{\cos\Phi}U\right) = 0. \end{aligned} \tag{109}$$

If the flow in transformed space is given by

$$V = v_g^*\frac{\mathcal{F}(\mathbf{x})^2L(\mathbf{X})}{\mathcal{F}(\mathbf{X})^2L(\mathbf{x})} \tag{110}$$

and

$$U = u_g^*\frac{\cos\Phi}{\cos\phi}, \tag{111}$$

then the identity (109) holds when the semi-geostrophic equations (6) are satisfied in physical space and $\mathcal{F} = f$.

It is appropriate to say a few words about the calculation leading up to the conclusions stated at the end of the last paragraph. In differentiating (101) along fluid trajectories, we have implicitly constructed a vector field

$$\frac{\mathrm{d}d}{\mathrm{d}t} = \frac{\partial d}{\partial t} + \frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial d}{\partial x_i} = \frac{\partial d}{\partial t} + \frac{\mathrm{d}X_i}{\mathrm{d}t}\frac{\partial d}{\partial X_i} \tag{112}$$

with components $\mathrm{d}x_i/\mathrm{d}t$ (resp. $\mathrm{d}X_i/\mathrm{d}t$) and with the set of basis vectors $\{\partial d/\partial x_i\}$ (resp. $\{\partial d/\partial X_i\}$). From this point of view, (109) is an identity, derived from (106), satisfied by a linear combination of the basis vectors

$$\{(-g/(a\mathcal{F}))\partial h/\partial\phi, (g/(a\cos\phi\mathcal{F}))\partial h/\partial\lambda\}.$$

This implies an additional constraint must be satisfied, namely that the coefficients vanish, which yields the result we sought to establish. A discussion of ideas related to this construction can be found in §4.25 of Schutz (1980).

Finally, we may write (110) and (111) in the following way that makes the geometry explicit. Substituting from (104) and (105) into the definition of the geostrophic wind (5), we can write (110) as

$$V = \frac{g}{a\cos\phi}\frac{\partial H}{\partial\Lambda}\frac{\mathcal{F}(\mathbf{x})L(\mathbf{X})}{\mathcal{F}(\mathbf{X})^2L(\mathbf{x})}, \tag{113}$$

and (111) becomes

$$U = -\frac{g}{a}\frac{\partial H}{\partial\Phi}\frac{\cos\Phi}{\cos\phi}\frac{L(\mathbf{X})\mathcal{F}(\mathbf{x})}{L(\mathbf{x})\mathcal{F}(\mathbf{X})^2}. \tag{114}$$

Now recall that, from (93), $L(\mathbf{x}) = \mathcal{F}(\mathbf{x})(1+\cos^2\phi(\mathrm{d}\lambda/\mathrm{d}\phi)^2)^{\frac{1}{2}}$ and we define $L(\mathbf{X}) = \mathcal{F}(\mathbf{X})(1+\cos^2\Phi(\mathrm{d}\Lambda/\mathrm{d}\Phi)^2)^{\frac{1}{2}}$, and therefore we can write the equations above as

$$V = \frac{g}{a\cos\phi\mathcal{F}(\mathbf{X})}\frac{\partial H}{\partial\Lambda}\frac{(1+\cos^2\Phi(\mathrm{d}\Lambda/\mathrm{d}\Phi)^2)^{\frac{1}{2}}}{(1+\cos^2\phi(\mathrm{d}\lambda/\mathrm{d}\phi)^2)^{\frac{1}{2}}}, \tag{115}$$

and

$$U = \frac{-g}{a\mathcal{F}(\mathbf{X})}\frac{\partial H}{\partial\Phi}\frac{\cos\Phi}{\cos\phi}\frac{(1+\cos^2\Phi(\mathrm{d}\Lambda/\mathrm{d}\Phi)^2)^{\frac{1}{2}}}{(1+\cos^2\phi(\mathrm{d}\lambda/\mathrm{d}\phi)^2)^{\frac{1}{2}}}, \tag{116}$$

respectively.

4.3 Solution of the transformed equations

In the previous subsection, we have shown that the semi-geostrophic equations on the sphere transform into the equations $a\cos\Phi\dot{\Lambda} = U, a\dot{\Phi} = V$, with (U,V) given by (113, 114), under the transformation from physical coordinates to the new coordinates (Λ,Φ); as long as $\mathcal{F} = f$. As in the f-plane theory, we regard $\mathbf{U} = (U,V)$ as a velocity field in a phase space with coordinates (Λ,Φ), the space being a copy of the original sphere S^2. Since σ is a measure of mass in phase space, standard kinematics yields the conservation law

$$\frac{\partial\sigma}{\partial t} = \nabla\cdot(\sigma\mathbf{U}). \tag{117}$$

The continuity equation (2) implies that, within any material circuit defined by fixed values of the lagrangian coordinates α and β:

$$\int h(\alpha,\beta,0)\mathrm{d}\alpha\mathrm{d}\beta = a\int h(\alpha,\beta,t)\mathrm{d}\mu = a\int \sigma(\alpha,\beta,t)\mathrm{d}\nu \tag{118}$$

This takes the form of a conservation of 'circulation' in phase space, though it has nothing to do with the circulation in physical space. It is a semi-geostrophic analogy of the 'impermeability' result of Haynes and McIntyre (1990).

Equation (117) would imply lagrangian conservation of σ if $\mathbf{U}$ were non-divergent. If $\mathbf{U}$ were exactly a 'geostrophic' wind in (Λ,Φ) space, then $\mathcal{F}(\mathbf{X})\mathbf{U}$ would be non-divergent. In fact, the divergence of $\mathcal{F}(\mathbf{X})\mathbf{U}$ is

$$g\nabla G\times\nabla H \tag{119}$$

where

$$G=\frac{\cos\Phi}{\cos\phi}\frac{L(\mathbf{X})\mathcal{F}(\mathbf{x})}{L(\mathbf{x})\mathcal{F}(\mathbf{X})}; \tag{120}$$

and H is defined in equation (101). The function G is the product of three ratios which tend to 1 as $d^2(\mathbf{x},\mathbf{X})$ tends to zero. As written in (120), G appears to have a singularity when $\mathbf{x}$ is at the pole and $\cos\phi=0$. This is an artefact resulting from the singularity of the coordinate system. It also appears to have a singularity if the geodesic connecting $\mathbf{x}$ to $\mathbf{X}$ is parallel to a line of latitude at $\mathbf{X}$, so that $\mathrm{d}\Lambda/\mathrm{d}\Phi$ is infinite. This is an artefact resulting from the use of $\tau=\phi$ as the path parameter in section 4.1. The remarks following (54) show that L is always bounded above and away from zero. The ratios in (120) depart from 1 because of curvature of the path, and are significantly different from 1 if the angle in euclidean 3-space between the directions of the path at $\mathbf{x}$ and $\mathbf{X}$ is significant. This occurs if $\mathbf{x}$ and $\mathbf{X}$ are separated by a distance comparable to the radius of the Earth, and in particular that the latitude is significantly different at the ends. The divergence (119) is only large if the ratios vary rapidly along contours of H, i.e. along the direction of $\mathbf{U}$. For the data shown in Fig.1, maximum values of $\mathbf{u}_g$ are about $15\mathrm{ms}^{-1}$. Thus $d(\mathbf{x},\mathbf{X})$ has values up to about 1.5×10^5m. The ratios that appear in G will then all depart from unity by about 1%, so overall G will depart from unity by up to 5%. The geostrophic wind $\mathbf{u}_g$ varies along h contours on a length scale of about 10^6m. Thus ∇G is of order 5×10^{-8}, so that $\nabla\cdot\mathcal{F}(\mathbf{X})\mathbf{U}$ is of order $5\times10^{-12}\mathbf{U}$. This has to be compared with a typical gradient of $\mathcal{F}(\mathbf{X})\mathbf{U}$, which is of order $10^{-10}\mathbf{U}$. The loss of lagrangian conservation of $\mathcal{F}(\mathbf{X})\mathbf{U}$, given by $\sigma\nabla\cdot(\mathcal{F}(\mathbf{X})\mathbf{U})$, is only a few percent of the lagrangian transport term $\mathcal{F}\mathbf{U}\cdot\nabla\sigma$. Note also that, in the special case of zonal geostrophic flow, $G\neq1$ but (119) vanishes because H and G are both independent of Λ.

As in Cullen and Gangbo (2001), the solution procedure is now to specify $\sigma(\Lambda,\Phi)$ at $t=0$, to solve the optimal map problem to find $\mathbf{x}=(\lambda,\phi)$ as a function of $\mathbf{X}=(\Lambda,\Phi)$, which also determines h and H, and then to calculate U and V from (113) and (114). The conservation law (117) can then be integrated in time, in principle, to give σ at later times. Formal analysis of this claim is outside the scope of this paper. However, we note the following facts:
i) U and V are bounded as discussed above.
ii) The difference between h and a locally convex function is the smooth non-oscillatory function $d^2(\mathbf{x},\mathbf{X})$. Thus we can expect that the property that the derivatives of a sequence of convex functions converge to the derivative of the limit convex function in a weak sense will also hold for involutive functions.

In the solution procedure of Cullen and Gangbo (2001), the velocities U, V were replaced by smoothed velocity fields U_γ, V_γ, so that the equation (117) could be solved, giving a solution h_γ. They then proved convergence of the sequence of approximations h_γ to a limit (which solves the original problem in a generalised sense). In the present case, we expect the regularity following from the involutive property of $-h$ to be sufficient to prove convergence.

A similar procedure can then be used to deal with the equator. The regularised problem that we have discussed has solutions h_ϵ, with geostrophic winds defined by using (5) with a positive value of the modified Coriolis parameter $\mathcal{F}$ in each hemisphere. We now let ϵ tend to zero. The involutive condition on $-h$ means, in particular, that $\frac{1}{a^2\cos^2\phi}\frac{\partial}{\partial\lambda}\left(\mathcal{F}^{-1}\frac{\partial h_\epsilon}{\partial\lambda}\right) > -\mathcal{F}$ and so $\mathcal{F}^{-1}\frac{\partial h_\epsilon}{\partial\lambda}$ and hence v_g must tend to zero at $\phi = 0$ as ϵ tends to zero. Since H is also involutive, V must tend to zero at $\Phi = 0$ as ϵ tends to zero. The geostrophic winds in the regularised problem are calculated from h by using a positive modified Coriolis parameter $\mathcal{F}$. The geostrophic winds in the limit solution are thus those given by setting $f = 2\Omega|\sin\phi|$. Since semi-geostrophic solutions are not preserved if f is replaced by $-f$, the limit solution will not be correct in the Southern hemisphere, so we must modify our solution procedure.

Semi-geostrophic solutions are invariant to a transformation which leaves h fixed, reverses the signs of f, $\partial/\partial\phi$, v and v_g, and leaves u, u_g and $\partial/\partial\lambda$ fixed. Solutions of equation (117), along with (113) and (114), are preserved if H and the coordinates are fixed, but U, V, f and F change sign. Thus we solve (117), but with the U and V in (113) and (114) multiplied by a factor $\alpha(\Phi)$, such that $\alpha = 1$ for $\Phi > \eta$, $\alpha = -1$ for $\Phi < -\eta$, with a smooth variation in between. This will give a solution for h which satisfies the semi-geostrophic equations on the sphere with the correct definition of f in the limit as $\eta \to 0$. Since $V = 0$ at $\Phi = 0$ in the limit solution, the limit solution will not have a singular $\nabla \cdot \mathbf{U}$ at $\Phi = 0$.

We finally demonstrate that the transformed equations (113) and (114) are consistent with the energetics that can be derived from the original equations (3), (5), (6). Using the passive variable property $\partial H/\partial t = \partial h/\partial t$ and the fact that, from (113) and (114) it is easy to show that

$$(1/(a\cos\Phi)(\partial H/\partial\lambda)U + (1/a)(\partial H/\partial\Phi)V = 0, \tag{121}$$

it follows that $\frac{DH}{Dt} = \frac{\partial h}{\partial t}$. Using (3), (5), (6), we can derive

$$\frac{D}{Dt}(\tfrac{1}{2}(u_g^2 + v_g^2)) = -\mathbf{u}\cdot\nabla h. \tag{122}$$

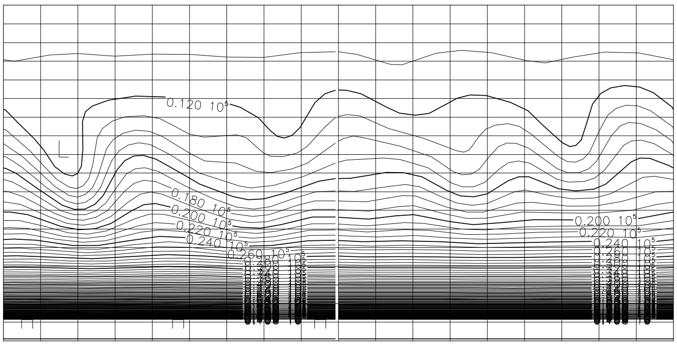
Combining these results shows that

$$\frac{D}{Dt}(\tfrac{1}{2}(u_g^2 + v_g^2) + h - H) = 0. \tag{123}$$

The quantity within the material derivative is, however, identically zero because of (101), demonstrating the consistency of our derivation.

4.4 Examples of the solution

In this subsection we show an example of the evolution of the semi-geostrophic model using the initial conditions illustrated in Figure 3. We plot the approximately conserved quan-



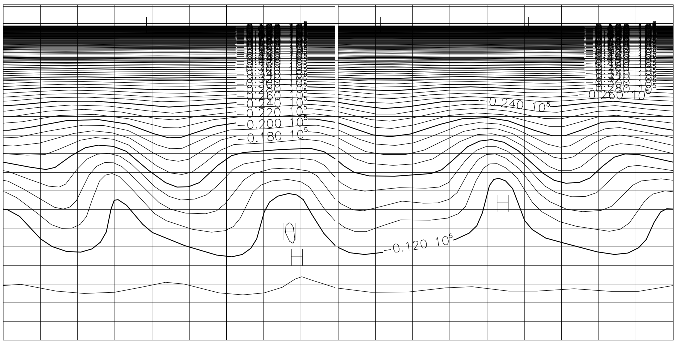


Figure 4: Distribution of $\sigma/(h_0 f(\Phi))$ derived from data shown in Figure 3. Contour interval 10^4s. Top: Northern hemisphere, bottom: Southern hemisphere.

tity $\sigma/(h_0 f(\Phi))$. Figure 4 shows the initial data corresponding to Figure 3. Values close to the equator are not plotted. The L_2 norm of $\sigma/(fh_0)$ is infinite, so instead we calculate the L_2 norm of the determinant of the matrix (24) divided by fh. This is the physical space form of the semi-geostrophic potential vorticity, calculated using the local value of f.

The fields of σ/h_0 and $\sigma/(fh_0)$ are shown in Figures 5 and 6 after 2 and 20 days integration respectively. The L_2 norm of the physical space potential vorticity is increased by a factor of 1.0006 after 2 days, and by 1.0029 after 20 days. Since the numerical methods used do not conserve potential vorticity exactly in the f-plane case, it would be difficult to determine whether potential vorticity is conserved or not from this diagnostic. The worst case estimate made in the previous subsection is almost certainly an overestimate of the effect on a global integral. Comparison of Figs 3 and 5 shows that the disturbances propagate faster near the equator, as expected from the dispersion formula for Rossby waves in spherical semi-geostrophic theory which is identical to that derived from the primitive shallow water equations, (Mawson (1996), p.280). After 20 days the disturbances have migrated closer to the equator, thus providing a severe test for the integration scheme.

5 Issues for further work

We have shown how the geostrophic coordinate transformation can be extended to the sphere in a well-defined way. The convexity principle which makes the coordinate transformation well-defined in the f-plane case has a natural generalisation. If a potential vorticity or density is defined respectively as the mass-weighted Jacobian of the forward and reverse transformation, we have shown how a potential density inversion procedure can be implemented. The evolution equations take the form of the transport of potential density by a 'velocity' parallel to the geostrophic wind. We have shown formally how the resulting equations can be solved.

There are several remaining analytic issues to be resolved in order to make the above solution procedure rigorous. The results of McCann (2001) and Cullen and Gangbo (2001) have to be extended to 'measure'-valued potential densities. This is because we cannot guarantee that the divergent part of the velocity in transformed coordinates will not lead to local concentration of potential density. In addition, we have to prove that the limit of the regularised semi-geostrophic solutions on the sphere is well-defined as the regularisation parameters ϵ, η tend to zero. Since the proofs that solutions exist rely on convergence of a sequence of approximations to the depth field, and the depth field is very flat near the equator, this is likely to be straightforward.

Meteorologically, the constraints on the dynamical system resulting from the conservation laws need to be explored, in particular to establish which flows are nonlinearly stable under this type of dynamics. An important issue is the apparent lack of an equivalent to the absolute vorticity conservation law satisfied by the barotropic vorticity equation on the sphere. This may be related to the need to enforce inertial stability. Inertial stability is harmless in the barotropic vorticity equation. The inertial stability condition prevents the model describing genuinely two-dimensional disturbances to the depth field near the equator. This may reflect the clearly different dynamics that is observed near the equator, for instance the inability of tropical cyclones to form close to the equator.

6 Acknowledgements

The first version of this work was carried out at the Isaac Newton Institute programme on 'Atmosphere-Ocean Dynamics' held from July to December 1996. The main part of the work was carried out under EPSRC grant no. GR/L43220. MJPC wishes to thank Robert McCann, Mikhail Feldman and Bo Su who provided useful input during a visit to Madison, Wisconsin, funded by NSF grant no. DMS 00 74 037 and also Cedric Villani who provided useful input during a visit to Reading in January 2003.

7 Appendix

Theorem 6 *Let $\mathcal{S}$ be a curved surface lying in a three-dimensional space, having unit normal $\mathbf{N}$, and bounded by a closed curve $\mathcal{C}$ whose unit normal locally tangent to the surface is $\mathbf{n}$. Let $\mathbf{B}$ be a vector field which, on $\mathcal{S}$, is tangent to $\mathcal{S}$. Then*

$$\int_{\mathcal{S}} [\mathrm{div}\mathbf{B} - \mathbf{N}.(\mathbf{N}.\mathrm{grad})\mathbf{B}]\, \mathrm{d}S = \oint_{\mathcal{C}} \mathbf{n}.\mathbf{B}\mathrm{d}s \tag{124}$$

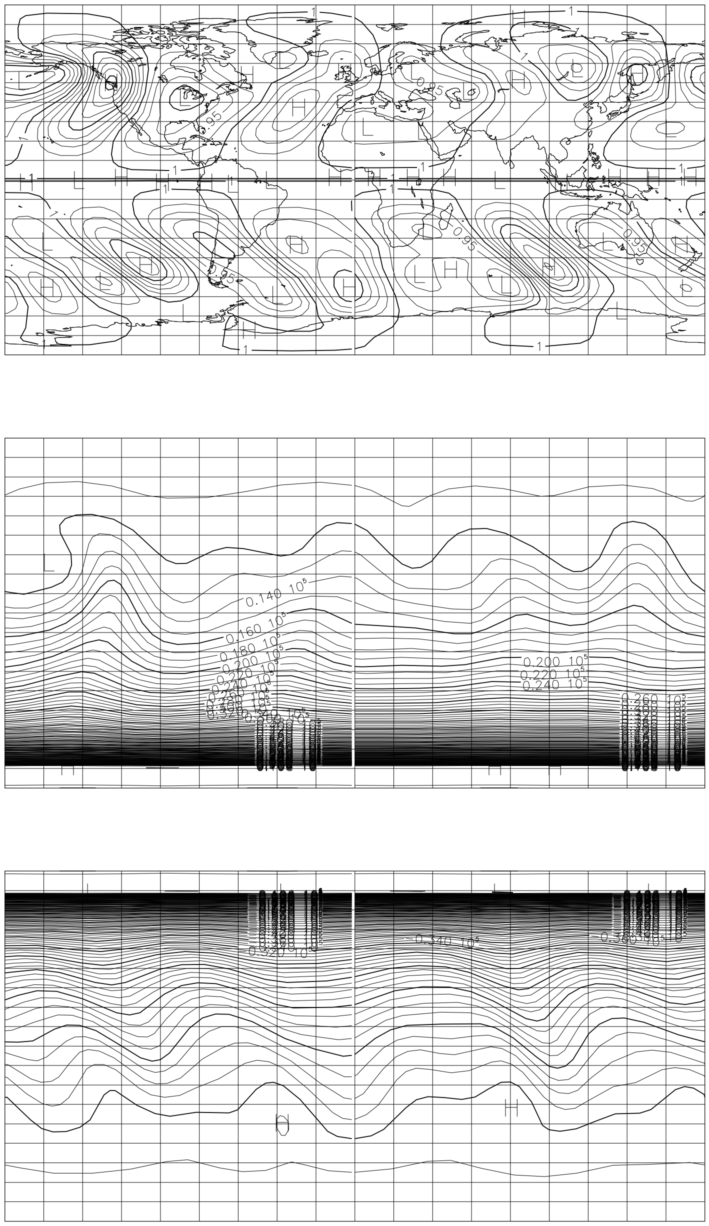


Figure 5: Top: distribution of $\sigma/(h_0)$ after 2 day forecast from initial data shown in Fig.3. Contour interval 0.025. Middle: distribution of $\sigma/(h_0 f(\Phi))$ at the same time (Northern hemisphere. Contour interval 10^4s. Bottom: as middle picture for Southern hemisphere.

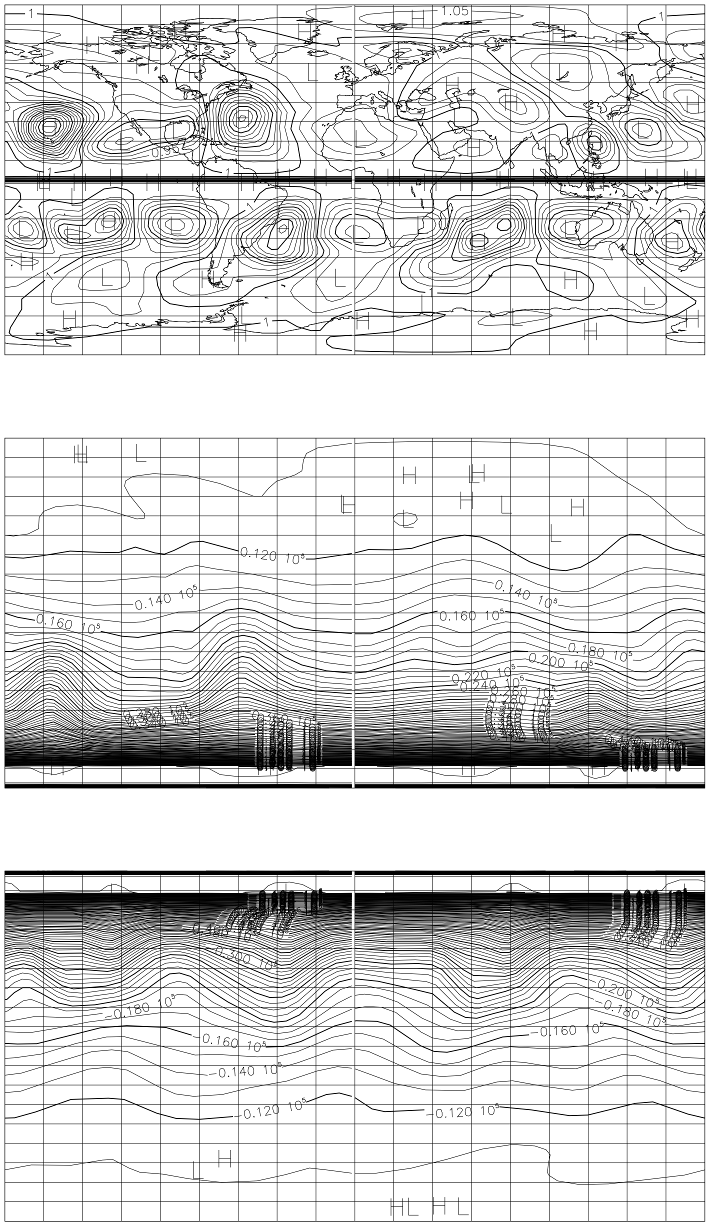


Figure 6: Top: distribution of $\sigma/(h_0)$ after 20 day forecast from initial data shown in Fig.3. Contour interval 0.025. Middle: distribution of $\sigma/(h_0 f(\Phi))$ at the same time (Northern hemisphere. Contour interval 10^4s. Bottom: as middle picture for Southern hemisphere.

where ds *is the measure of distance along* $\mathcal{C}$, *and* div *and* grad *are the vector differential operators in three-dimensional space.*

Proof

This is a corollary of Stokes' Theorem, which states that

$$\int_{\mathcal{S}} \mathbf{N}.(\mathrm{curl}\mathbf{A})\mathrm{d}S = \oint_{\mathcal{C}} \mathbf{A}.\mathbf{t}\mathrm{d}s \tag{125}$$

for any three-dimensional smooth vector field $\mathbf{A}$, where $\mathbf{t}$ is the local unit tangent to $\mathcal{C}$ and curl is the vector differential operator in three dimensions.

First choose *any* smooth vector field $\mathbf{B}$, and construct $\mathbf{A} = \mathbf{N} \times \mathbf{B}$ on $\mathcal{S}$. Then on $\mathcal{C}$

$$\mathbf{t}.\mathbf{A} = \mathbf{t}.(\mathbf{N} \times \mathbf{B}) = (\mathbf{t} \times \mathbf{N}).\mathbf{B} = \mathbf{n}.\mathbf{B},$$

by defining $\mathbf{n} = \mathbf{t} \times \mathbf{N}$. Hence

$$\oint_{\mathcal{C}} \mathbf{A}.\mathbf{t}\mathrm{d}s = \oint_{\mathcal{C}} \mathbf{n}.\mathbf{B}\mathrm{d}s$$

which is the required form of the right hand side of (124). (The application to (8) will require $\mathbf{B}$ to be, on $\mathcal{S}$, $\mathbf{B} = h^2\dot{\mathbf{r}}$.) Second invoke the vector analysis identity

$$\mathrm{curl}(\mathbf{N} \times \mathbf{B}) = (\mathbf{B}.\mathrm{grad})\mathbf{N} - \mathbf{B}(\mathrm{div}\mathbf{N}) - (\mathbf{N}.\mathrm{grad})\mathbf{B} + \mathbf{N}(\mathrm{div}\mathbf{B}),$$

then on $\mathcal{S}$

$$\mathbf{N}.[\mathrm{curl}(\mathbf{N} \times \mathbf{B})] = \mathbf{N}.(\mathbf{B}.\mathrm{grad})\mathbf{N} - \mathbf{N}.\mathbf{B}(\mathrm{div}\mathbf{N}) - \mathbf{N}.(\mathbf{N}.\mathrm{grad})\mathbf{B} + (\mathrm{div}\mathbf{B})$$

because $\mathbf{N}.\mathbf{N} = 1$. Now invoke the properties that $\mathbf{B}$ is tangential to the surface $\mathcal{S}$, so that $\mathbf{N}.\mathbf{B} = 0$, and

$$\mathbf{N}.(\mathbf{B}.\mathrm{grad})\mathbf{N} = \mathbf{N}.\frac{\partial \mathbf{N}}{\partial B} = \tfrac{1}{2}\frac{\partial \mathbf{N}.\mathbf{N}}{\partial B} = 0,$$

because $\mathbf{N}.\mathbf{N} = 1$ and $\mathbf{B}.\mathrm{grad} = \partial/\partial B$ is the derivative in the direction of $\mathbf{B}$. Thus

$$\int_{\mathcal{S}} [\mathrm{div}\mathbf{B} - \mathbf{N}.(\mathbf{N}.\mathrm{grad})\mathbf{B}]\mathrm{d}S = \oint_{\mathcal{C}} \mathbf{n}.\mathbf{B}\mathrm{d}s \tag{126}$$

and because the differential operators are defined in three-dimensional space, $\mathbf{N}(\mathbf{N}.\mathrm{grad})$ is the gradient in the $\mathbf{N}$-direction, therefore grad $-$ $\mathbf{N}(\mathbf{N}.\mathrm{grad})$ is the gradient tangential to the surface. Q.E.D. □

The application of this result to (8) requires $\mathbf{B} = h^2\dot{\mathbf{r}}$ and then we obtain the required result

$$\int_{\mathcal{S}} [\mathrm{div}(h^2\dot{\mathbf{r}}) - \mathbf{N}.(\mathbf{N}.\mathrm{grad})h^2\dot{\mathbf{r}}]\mathrm{d}S = \oint_{\mathcal{C}} h^2\mathbf{n}.\dot{\mathbf{r}}\mathrm{d}s.$$

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