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Families of Contact Transformations

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Abstract

An investigation of the contact transformation is carried out, mainly in the lowest dimensional case. This case is chosen because it allows a very general family of explicit contact transformations to be derived, which is not available elsewhere. This family gives a desirable perspective to certain familiar particular cases which it includes, associated with the Legendre transformation and with the semi—geostrophic equations of meteorology. It also displays a clear distinction between contact and canonical transformations, which have often been confused in the past. For clarity we begin by introducing the concept of a lift transformation. A brief study of generalized duality is also included. Contact transformations have an important role in the study of differential equations.

1. Introduction

The aim of this paper is to convey a very explicit understanding of the contact transformation. The subject of contact geometry has a long history, and some modern writing gives the impression of a high level of sophistication, but without always being accessible or precise. Our study of the literature has convinced us that any potential user will have the need for a significant body of concrete examples of contact transformations, and this paper provides that.

In a previous paper (Sewell and Roulstone, 1993) we gave new examples of the canonical transformation, and compared several distinct definitions of it, which are all different from our definition here of the contact transformation. One of the remarkable features of the history of mathematical physics is the long standing confusion between the two transformations. It is interesting to notice, for example, that in his first edition Goldstein (1950, p. 239) treated them as synonymous, whereas in his second edition (1980, p. 382) he has realized that there is a difference between them.

We study the contact of plane curves in terms of the intersection of their lifts. We specify the distinction between regular and singular lifts. The latter include a vertical segment whose projection is a single fixed point, and we give examples showing that such a lift is far from being a pathological case, but completes the theory of contact in a natural way. This is related to the fact that the envelope of a family of curves which all pass through a fixed point must include that point, and so may consist of disjoint parts. For clarity we make explicit the distinction between a lift transformation and a contact transformation. This puts us into position to prove Theorem 3, which is the central result in the paper. This establishes a much wider family of lift and contact transformations than has been demonstrated before, as far as we are aware, and we extend some of these to any odd number of dimensions. We show how Theorem 3 includes two very special cases, associated with the Legendre transformation and with the geostrophic transformation of meteorology. The geostrophic transformation was in fact the starting point which showed us the need for this investigation and the previous one. We prove that the lifts of the plane curves which participate in the Legendre transformation are converted into each other by the associated lift transformation, and in Theorem 10 we generalize this result in a study of generalized duality which links with other parts of the literature.

2. Lifted curves and the contact of plane curves

We need to begin with a critical review of some ideas in contact geometry, in the simplest possible context, to establish a basis for subsequent Sections.

Let x, y, z be cartesian coordinates in R^3. A plane curve in the x, y plane is given in parametric form by the relations

$$x = x(u) \quad , \quad y = y(u) \tag{1}$$

between the coordinates and given functions, on the right, of a scalar parameter u for some continuous range of u. When $z(u)$ is any given function, the plane curve (1) can be regarded as the <u>projection</u> of the space curve

$$x = x(u) \quad , \quad y = y(u) \quad , \quad z = z(u) \tag{2}$$

where $x(u)$ and $y(u)$ are the same as in (1). One might say that every such curve (2) is a lifted version of (1) into R^3.

For the purposes of contact geometry, however, it is conventional to adopt a narrower meaning of this terminology. In this paper we follow that convention to the extent of adopting the definition that (2) is a <u>lifted curve</u>, or <u>lift</u> of (1), if and only if $z(u)$ satisfies

$$\frac{dy}{du} = z\,\frac{dx}{du}\,. \tag{3}$$

Given the plane curve (1), however, this formula (3) does not always fully specify the function $z(u)$. To elucidate this uncertainty, we define each point of the plane curve to be

$$\underline{\text{regular}} \text{ if } \frac{dx}{du} \neq 0 \quad \text{or} \quad \underline{\text{singular}} \text{ if } \frac{dx}{du} = 0\,. \tag{4}$$

At a regular point the inverse function theorem guarantees that we can use x as the parameter and write (1) with (3) as

$$y = y(x) \quad \text{with gradient} \quad z = \frac{dy}{dx}. \tag{5}$$

Then the ordinate of the lift is the gradient of the plane curve, and we can say that the lift has a regular point on it too. Two different lifts which intersect at regular points therefore have projections which are in <u>contact</u>, in the sense that they meet and have the same tangent line there. For example, $y = \frac{1}{3}x^3$ and $y = x - \frac{2}{3}$ are in contact at $x = 1$, $y = \frac{1}{3}$ where $dy/dx = 1$ for both, and their lifts have a regular point in common there.

It proves very convenient to discuss the contact of plane curves in terms of the intersection of their lifts. To do this with reasonable generality, however, we must address the fact that at a singular point, or <u>singularity</u> of the plane curve (1), equation (3) gives incomplete information about the value of z on the lift. A distinction between two types of singular point must be made before we can decide what further information should be added to (3) to complete the description of the lift.

We call the singularity <u>isolated</u> if $dx/du = 0$ at an isolated value of u, so that adjacent points on the plane curve are regular ones. If $dy/du \neq 0 = dx/du$, (3) implies that z must be infinite. However, if $dy/du = 0 = dx/du$, (3) implies nothing about z, but is certainly satisfied for any finite z. When $x(u)$ and $y(u)$ are analytic functions, we can define $z(u)$ in (2) by

$$z(u) = \lim_{\epsilon \to 0} \frac{\dfrac{dy}{du} + \dfrac{1}{2}\dfrac{d^2y}{du^2}\epsilon + \dots}{\dfrac{dx}{du} + \dfrac{1}{2}\dfrac{d^2x}{du^2}\epsilon + \dots} \tag{6}$$

where ϵ is the u–increment, and the derivatives are evaluated at $\epsilon = 0$. At regular points (3) and (6) are equivalent. At isolated singular points (6) is consistent with (3), but (3) does not imply (6). We adopt (6) because it gives the interpretation that the ordinate of the lift is the gradient of the plane curve at such isolated singularities (as well as at regular points), and we can say that the lift has an isolated singularity on it there too. For example, if $dy/du = dx/du = 0 \neq d^2x/du^2$, (6) specifies the value

$$z = \frac{d^2y/du^2}{d^2x/du^2} \tag{7}$$

on the lift. Two different lifts which intersect at a regular point of one and at an isolated singularity of the other, or at an isolated singularity of each, therefore have projections which are in contact, because (6) ensures that the projections have the same tangent line where they meet.

A singularity where $dx/du \equiv 0$, i.e. where x is constant, is not isolated. Then (1) could be either a straight line parallel to the y–axis if $dy/du \not\equiv 0$, or a fixed point if $dy/du \equiv 0$, i.e. where y is constant. A lift of the first option would require z to be infinite to satisfy (3). The second option is of more interest. A fixed point in the x, y plane satisfies (3) for any range of finite z. Therefore we define the lift of a <u>fixed point</u> to be any finite line segment vertically above it, which we can call a <u>vertical singularity</u>.

A fixed point in the x, y plane will be deemed to be in contact with another plane curve if that fixed point lies on the curve. This type of contact is equivalent to the intersection of the vertical singularity with the lift, at a regular point or an isolated singular point, of the plane curve, i.e. in the same terms as the ideas already described. This is a recognised convention, although it is expressed in other ways in the literature (e.g. see Fig. 15.6 of Burke, 1985).

An example of a lifted curve containing only regular points is

$$x = \frac{u}{(1+m^2)^{\frac{1}{2}}}\ ,\quad y = \frac{u\,m}{(1+m^2)^{\frac{1}{2}}} - \frac{1}{2}u^2\ ,\quad z = m - u(1+m^2)^{\frac{1}{2}} \tag{8}$$

with any fixed m, which we shall use subsequently.

The curve

$$x = \frac{1}{2}u^2 - mu \quad , \quad y = \frac{1}{3}u^3 - \frac{1}{2}mu^2 \quad , \quad z = u \tag{9}$$

with any fixed m, satisfies (6). It illustrates a lifted curve which has an isolated singularity, at $u = m$. We can use z as the parameter everywhere, but x is only available as an alternative parameter for two–sided limits where $x > -\frac{1}{2}m^2$, and not through the singularity itself.

3. Lift transformations and contact transformations

We now consider a mapping

$$X = X(x, y, z) \quad , \quad Y = Y(x, y, z) \quad , \quad Z = Z(x, y, z) \tag{10}$$

of $R^3 \longrightarrow R^3$. Here X, Y, Z are the cartesian coordinates in the second R^3. The functions of x, y, z on the right are assumed to be single valued. The first two functions are also assumed to have single valued derivatives. The mapping will transform any space curve (2), whether it is a lift or not, into another space curve

$$X = X(u) \quad , \quad Y = Y(u) \quad , \quad Z = Z(u) \tag{11}$$

in the second R^3, where functions of u are on the right. This (11) will be a lift of the plane curve

$$X = X(u) \quad , \quad Y = Y(u) \tag{12}$$

if and only if

$$\frac{dY}{du} = Z\,\frac{dX}{du}\,, \tag{13}$$

and reasoning like that associated with (4) – (9) again applies.

A lift transformation is defined to be a mapping (10) which converts one lift satisfying (3) into another lift satisfying (13). It will be one of the following two types.

A general lift transformation converts every lift into another lift, and does not depend on the curve being transformed.

A special lift transformation converts only a restricted class of lifts into other lifts, and has a dependence upon that class of lifts.

It may be expected that general lift transformations are the more important type, and Theorem 3 is the central result of this paper which describes a large family of them. Some special lift transformations are readily obtainable, however, and we illustrate how to do that in the last part of Theorem 2 and in §6.

A contact transformation is a lift transformation which applies to a pair of lifts which intersect in the first R^3. It therefore transforms contact in the associated first plane into contact in the second plane.

Therefore to be sure that a lift transformation is also a contact transformation one really needs to either exhibit a pair of plane curves in contact, or nominate a fixed point on the plane curve. The transformation then applies to their lifts. The second option is always available, but the first needs to be demonstrated. If it is assumed, as may be common, then the distinction between a lift transformation and a contact transformation becomes blurred.

The envelope of a class of plane curves is a familiar method of illustrating contact, in this case between the envelope and any member of the class. The envelope itself is often not a member of the class, however, and this may mean that a general but not a special lift transformation will serve as a contact transformation in such a case.

A prerequisite for the study of contact transformations defined in this way is evidently an analysis of lift transformations.

For this purpose it will be convenient to use the following shorthand notation for any mapping (10).

$$\alpha = \frac{\partial Y}{\partial x} - Z\frac{\partial X}{\partial x} \quad , \quad \beta = \frac{\partial Y}{\partial y} - Z\frac{\partial X}{\partial y} \quad , \quad \gamma = \frac{\partial Y}{\partial z} - Z\frac{\partial X}{\partial z} . \qquad (14)$$

Theorem 1

Necessary and sufficient conditions for (10) to have the property

$$\frac{dY}{du} - Z\,\frac{dX}{du} \;=\; \beta\left[\frac{dy}{du} - z\,\frac{dx}{du}\right] \tag{15}$$

for every smooth space curve (2) are that

$$\alpha + \beta z \;=\; 0 \quad , \quad \gamma \;=\; 0\,. \tag{16}$$

Then

$$Z \;=\; \frac{\dfrac{\partial Y}{\partial x} \;+\; z\dfrac{\partial Y}{\partial y}}{\dfrac{\partial X}{\partial x} \;+\; z\dfrac{\partial X}{\partial y}} \quad \text{and} \quad Z \;=\; \frac{\dfrac{\partial Y}{\partial z}}{\dfrac{\partial X}{\partial z}} \tag{17}$$

when $\dfrac{\partial X}{\partial x} + z\,\dfrac{\partial X}{\partial y} \neq 0$ and $\dfrac{\partial X}{\partial z} \neq 0$ respectively.

Proof

The chain rule shows that

$$\begin{aligned}\frac{dY}{du} - Z\,\frac{dX}{du} \;&=\; \alpha\,\frac{dx}{du} + \beta\,\frac{dy}{du} + \gamma\,\frac{dz}{du}\\ &=\; \beta\left[\frac{dy}{du} - z\,\frac{dx}{du}\right] + (\alpha + \beta z)\,\frac{dx}{du} + \gamma\,\frac{dz}{du}\,.\end{aligned} \tag{18}$$

The sufficiency of (16) is then immediate, and so is necessity if the tangent components dx/du and dz/du of the space curve are to be arbitrary. When (16) holds, (17) is an immediate consequence. □

Theorem 2

When (16) holds, (10) transforms any lift into another lift if

$$\beta \neq 0\,. \tag{19}$$

In other words, necessary and sufficient conditions for (10) to be a general lift transformation are that (16) and (19) hold.

A necessary and sufficient condition for a vertical singularity to transform into another lift is that $\gamma = 0$.

Proof

On any lift in the first R^3 (3) holds. From (15), (19) implies (13).

The case $\beta = 0$ in (16) is of no interest because then every curve in the first R^3 implies (13).

The last part follows from (18) because a vertical singularity has $dx/du \equiv 0$, $dy/du \equiv 0$ and $dz/du \equiv 1$, so that $(16)_1$ and (19) are not necessary for the stated particular conclusion.

□

Theorems 1 and 2 do not, of themselves, say anything about contact. The single valuedness assumed for (10) is needed, when Theorem 2 holds, to guarantee that when a pair of lifts intersect in the first R^3, their transforms will also intersect in the second R^3. This invariance of intersection is needed to justify the definition of a contact transformation adopted by Carathéodory (1982, §119), namely that of a mapping (10) which satisfies (15) with (19). The formulae (17) gives an alternative and more explicit confirmation of the invariance of an intersection of lifts, at least as far as their height is concerned. Variants of (17) are familiar in the literature (e.g., see Jeffrey and Taniuti, 1964). None of these authors distinguish between regular and singular points of lifts, or between lift and contact transformations.

4. Families of general lift transformations

Theorem 3

The following are explicit illustrations of general lift transformations (10) when $\frac{\partial X}{\partial z} \neq 0$. Each example listed below is a family of transformations whose coefficients may be any given constants, subject to the stated inequalities.

(i)

$$X = \tfrac{1}{2}Ax^2 + \tfrac{1}{2}Cz^2 + zx + RAx + (CA - 1)y + Rz\,,$$

$$Y = \tfrac{1}{2}gAx^2 + \tfrac{1}{2}cz^2 + gzx + rAx + (cA - g)y + rz\,,$$

$$Z = \frac{gx + cz + r}{x + Cz + R}\,, \quad x + Cz + R \neq 0\,, \tag{20}$$

$$\beta = \frac{(c - gC)(Ax + z + RA) + (1 - CA)(r - gR)}{x + Cz + R}\,,$$

$c - gC$ and $(1 - CA)(r - gR)$ not both zero.

(ii) $$X = \tfrac{1}{2}G(G + Q)x^2 + \tfrac{1}{2}z^2 + Gzx + R(G + Q)x + Qy + Rz\,,$$

$$Y = \tfrac{1}{2}g(G + Q)x^2 + \tfrac{1}{2}cz^2 + gzx + r(G + Q)x + [c(G + Q) - g]y + rz\,,$$

$$Z = \frac{gx + cz + r}{Gx + z + R}\,, \quad Gx + z + R \neq 0\,, \tag{21}$$

$$\beta = \frac{(cG - g)[(G + Q)x + z + R] - Q(r - cR)}{Gx + z + R}\,,$$

$g - cG$ and $Q(r - cR)$ not both zero.

(iii) $$X = \tfrac{1}{2}GPx^2 + \tfrac{1}{2}Cz^2 + Gzx + Px + (CP - G)y + z\,,$$

$$Y = \tfrac{1}{2}gPx^2 + \tfrac{1}{2}cz^2 + gzx + rPx + (cP - g)y + rz\,,$$

$$Z = \frac{gx + cz + r}{Gx + Cz + 1}\,, \quad Gx + Cz + 1 \neq 0\,, \tag{22}$$

$$\beta = \frac{(Gc - Cg)(Px + z) - (g - rG) + P(c - rC)}{Gx + Cz + 1}\,,$$

$Cg - Gc$ and $g - rG - P(c - rC)$ not both zero.

Proof

We seek solutions of (16) among the families of quadratics

$$2X = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy + 2Px + 2Qy + 2Rz\ ,$$
$$2Y = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz\ . \tag{23}$$

These assumptions require, from (16), that when $Gx + Fy + Cz + R \neq 0$,

$$Z = \frac{gx + fy + cz + r}{Gx + Fy + Cz + R} \tag{24}$$

and

$$\begin{aligned}&(Gx + Fy + Cz + R)\,[(ax + hy + gz + p) + z(hx + by + fz + q)]\\ &= (gx + fy + cz + r)\,[(Ax + Hy + Gz + P) + z(Hx + By + Fz + Q)].\end{aligned} \tag{25}$$

Then

$$\begin{aligned}&(cF - Cf)z^3 + (gF - Gf + cH - Ch)xz^2 + (cB - Cb)gz^2\\
&+ (fH - Fh + gB - Gb)xyz + (gH - Gh)x^2z + (fB - Fb)y^2z\\
&+ (gA - Ga)x^2 + (fH - Fh)y^2 + [c(G + Q) - C(g + q) + rF - Rf]z^2\\
&+ [f(G + Q) - F(g + q) + cH - Ch + rB - Rb]yz\\
&+ (gQ - Gq + cA - Ca + rH - Rh)zx + (gH - Gh + fA - Fa)xy\\
&+ (gP - Gp + rA - Ra)x + (fP - Fp + rH - Rh)y\\
&+ [cP - Cp + r(G + Q) - R(g + q)]z + rP - Rp \ = \ 0\ .\end{aligned} \tag{26}$$

This equation can be satisfied identically by choosing all 16 coefficients in it to be zero. This provides relations between the 18 coefficients in the quadratics X and Y, but one in each of those can be regarded as a disposable scale factor. There are $9 \times 9 = 81$ different ways of fixing these scale factors, for example by choosing one coefficient in each of X and Y to be unity. This leaves 16 bilinear equations to be solved for 16 unknowns.

We provide sufficient detail that will allow the reader to reconstruct the proof and perceive its internal logic.

These 16 equations divide naturally into three groups. The first group, of nine equations, does not contain the coefficients P, Q, R, p, q, r of the linear terms in X and Y. It is

$$\begin{aligned} & gA - Ga = 0\,, \quad fA - Fa = 0\,, \\ & cB - Cb = 0\,, \quad gB - Gb = 0\,, \quad fB - Fb = 0\,, \\ & fH - hF = 0\,, \quad gH - hG = 0\,, \\ & cF - fC = 0\,, \quad gF - fG + cH - hC = 0\,. \end{aligned} \tag{27}$$

Equations (27) relate the ratios of the coefficients of the second degree terms in (23), to the extent that such ratios are well defined.

There is a second group of only three equations which contain P, R, p, r but not Q, q, namely

$$\begin{aligned} -Gp + gP + Ar - aR &= 0\,, \\ -Fp + fP + Hr - hR &= 0\,, \\ Pr - pR &= 0\,. \end{aligned} \tag{28}$$

The third group, of four equations, does contain Q and q, and is

$$
\left.
\begin{aligned}
&r(G+Q) - R(g+q) + cP - Cp = 0 , \\
&c(G+Q) - C(g+q) + rF - Rf = 0 , \\
&f(G+Q) - F(g+q) + rB - Rb + cH - Ch = 0 , \\
&g(G+Q) - G(g+q) + rH - Rh + cA - Ca = 0 .
\end{aligned}
\right\} \qquad (29)
$$

From (29) we see that Q and q only ever appear in conjunction with G and g respectively. In retrospect (29) would have been simpler if we had written $Q - G$ and $q - g$ in place of Q and q in (23).

The last seven of (27) and the last three of (29) imply

$$\beta = \frac{(cA - Ca)x + 2(cH - Ch)y + (cG - Cg)z + qR - Qr}{Gx + Fy + Cz + R} . \qquad (30)$$

We now prove that $F = 0$. Suppose that $F \neq 0$, and ensure this by choosing the scale factor in X to be $F = 1$. Then (27) implies

$$a = fA , \quad b = fB , \quad c = fC , \quad g = fG , \quad h = fH , \quad \beta = \frac{qR - Qr}{Gx + y + Cz + R} .$$

Then (28) implies

$$(A - GH)(r - fR) = 0 , \quad p - fP = H(r - fR) , \quad (P - RH)(r - fR) = 0 ,$$

and (29) implies

$$qR - Qr = (r - fR)(G - CH) ,$$

$$r - fR = C(q - fQ) , \quad q - fQ = B(r - fR) , \quad G(q - fQ) = H(r - fR) .$$

It follows that either $r = fR$, so that $\beta = 0$ immediately; or that $r \neq fR$ and $(q - fQ)(G - CH) = 0$, which implies $G = CH$ because $q \neq fQ$, so that $\beta = 0$ again. Hence

$$F = 0 \tag{31}$$

to ensure $\beta \neq 0$ as (19) requires.

The remainder of the proof divides naturally into three parts. We put $F = 0$ at the outset. Then, because $Gx + Cz + R \neq 0$, we need to explore the cases $G \neq 0$, $C \neq 0$ and $R \neq 0$ in turn.

(i) Suppose that $G \neq 0$, and ensure this by choosing the scale factor in X to be

$$G = 1 . \tag{32}$$

Then (27) implies

$$a = gA , \quad b = gB , \quad h = gH , \quad f = 0 ,$$

$$B(c - gC) = 0 , \quad H(c - gC) = 0 .$$

If we assume $c = gC$, $(28)_1$ and $(29)_1$ imply that $r \neq gR$ is needed to avoid $\beta = 0$, and then $(28)_2$ and $(29)_3$ imply $B = H = 0$. It is therefore more general to assume instead that (27) implies

$$B = H = 0 , \quad b = h = f = 0 , \quad a = gA .$$

Then (28) and (29) imply

$$p - gP = A(r - gR) , \quad q - gQ = A(c - gC) ,$$

$$(P - RA)(r - gR) = 0 , \quad (Q - CA + 1)(c - gC) = 0 ,$$

$$(P - RA)(c - gC) + (Q - CA + 1)(r - gR) = 0 ,$$

$$\beta = \frac{(c - gC)(Ax + z + RA) - Q(r - gR)}{x + Cz + R} .$$

Hence at most one of $c - gC$ and $r - gR$ can be zero to achieve $\beta \neq 0$, and both could be non–zero. In each case we deduce

$$P = RA , \quad Q = CA - 1 , \quad p = rA , \quad q = cA - g .$$

The family (20) results. The scale factor in Y could be chosen in several ways, depending on the use which may be required for the family; one way which ensures the validity of the inequalities stated in (19) is to choose

$$c - gC + (1 - CA)(r - gR) = 1 . \tag{33}$$

(ii) Suppose that $C \neq 0$, and ensure this by choosing the scale factor in X to be

$$C = 1 . \tag{34}$$

Then (27) implies

$$Ga = gA , \quad b = cB , \quad h = cH , \quad f = 0 ,$$

$$B(g - cG) = 0 , \quad H(g - cG) = 0 .$$

If we assume $g = cG$, $(28)_2$ with $(29)_2$ and $(29)_4$ imply that $r \neq cR$ is needed to avoid $\beta = 0$, and then $(28)_2$ and $(29)_3$ imply $B = H = 0$. It is therefore more general to assume instead that (27) implies

$$B = H = 0 , \quad b = h = f = 0 , \quad Ga = gA .$$

Then (28) and (29) imply

$$g - cG = cQ - q ,$$

$$p - cP = (G + Q)(r - cR) \quad , \quad a - cA = (G + Q)(cQ - q) ,$$

$$[P - R(G + Q)](r - cR) = 0 \quad , \quad [A - G(G + Q)](cQ - q) = 0 ,$$

$$[P - R(G + Q)](cQ - q) + [A - G(G + Q)](r - cR) = 0 ,$$

$$\beta = \frac{(cG - g)[(G + Q)x + z + R] - Q(r - cR)}{Gx + z + R} .$$

Hence at at most one of $g - cG$ and $r - cR$ can be zero to achieve $\beta \neq 0$, and both could be non–zero. In each case we deduce

$$A = G(G + Q) \quad , \quad P = R(G + Q) ,$$

$$a = g(G + Q) \quad , \quad p = r(G + Q) .$$

The family (21) results. The scale factor in Y can again be chosen in several ways; one way which ensures the validity of the inequalities stated in (21) is to choose

$$g - cG + Q(r - cR) = 1 . \qquad (35)$$

(iii) Suppose that $R \neq 0$, and ensure this by choosing the scale factor in X to be

$$R = 1 . \tag{36}$$

Putting $F = 0$ in (27) implies

$$gA = Ga , \quad gB = Gb , \quad gH = Gh , \quad cB = Cb , \quad cH = Ch , \tag{37}$$

$$fA = fB = fC = fG = fH = 0 . \tag{38}$$

There are apparently two cases here, but in reality only one. For if we assume that (38) implies

$$A = B = C = G = H = 0 , \tag{39}$$

then (28) and (29) imply

$$a = gP , \quad h = fP , \quad p = rP ,$$

$$f = cQ , \quad b = fQ , \quad h = gQ , \quad \beta = q - rQ = cP - g .$$

Hence $Q(cP - g) = 0$. Excluding $\beta = 0$ leaves $Q = 0$, but this implies $f = 0$. Therefore it is more general to assume that (38) implies

$$f = 0 \tag{40}$$

instead of (39), with (37) still available as it stands. Now (28) implies

$$p = rP \,, \quad h = rH \,, \quad a - rA = P(g - rG) \,,$$

and (29) with (37) then implies

$$b = rB \,, \quad B(g - rG) = H(g - rG) = B(c - rC) = H(c - rC) = 0$$

together with

$$g + q = r(G + Q) + P(c - rC) \,,$$

$$(G + Q - CP)(c - rC) = 0 \quad , \quad (A - GP)(g - rG) = 0 \,,$$

$$(G + Q - CP)(g - rG) + (A - GP)(c - rC) = 0 \,,$$

and

$$\beta = \frac{(rG - g)[(G + Q)x + Cz + 1] + (c - rC)[GPx + Gz + P]}{Gx + Cz + 1} \,.$$

We see that $g - rG$ and $c - rC$ cannot both be zero if $\beta = 0$ is to be avoided. It follows that

$$B = H = b = h = 0 \,, \quad G + Q = CP \,, \quad A = GP \,, \quad a = gP \,, \quad q = cP - g \,.$$

The family (22) results. Again the scale factor in Y can be chosen in several ways, and one way which ensures $\beta \neq 0$ is to choose

$$(C + 1)(g - rG) - (G + P)(c - rC) = 1 \,. \tag{41}$$

□

5. Examples

Theorem 4

The most general family of lift transformations possible in which Z is linear when X and Y could be quadratic is obtained by putting $G = C = 0$ in (22). Z cannot be constant with such X and Y, and it turns out that X must be linear.

Proof

This is immediate from Theorem 3(iii). □

For example, the last of

$$G = C = 0 \quad , \quad g = cP \pm 1 , \tag{42}$$

shows two possible alternative ways of fixing the previously arbitrary scale factor in Y. Then (22) becomes

$$\begin{aligned} X &= Px + z , \\ Y &= \tfrac{1}{2}(cP \pm 1)Px^2 + \tfrac{1}{2}cz^2 + (cP \pm 1)zx + rPx \mp y + rz , \\ Z &= (cP \pm 1)x + cz + r , \\ \beta &= \mp 1 , \end{aligned} \tag{43}$$

with arbitrary P, c and r.

All the lift transformations (43) have the especially simple property that any space curve (2) in x, y, z space is transformed into another space curve (11) in X, Y, Z space such that

$$\frac{dY}{du} - Z\frac{dX}{du} = \mp\left[\frac{dy}{du} - z\frac{dx}{du}\right] , \tag{44}$$

by (15). Whenever (2) or (11) is a lift, so is the other, by Theorem 2. Any pair of intersecting lifts is transformed into another pair of intersecting lifts (this is the contact transformation property of the associated plane curves or fixed points).

The special choice $P = c = r = 0$ with $g = +1$ in (43) gives $\beta = -1$ and

$$X = z, \quad Y = zx - y, \quad Z = x. \tag{45}$$

The first two of these equations have the property

$$Y + y = Xx \tag{46}$$

of a Legendre transformation which maps the point or pole x, y into the line or polar (46) in X, Y space, or a polar into a pole. The lift transformation (45) of $R^3 \to R^3$ therefore contains within it the Legendre transformation of $R^2 \to R^2$. The latter does not use $(45)_3$, so the two transformations are not equivalent. The last two of (45) have the property $Y + y = Zz$ of another Legendre transformation, which does not use $(45)_1$.

Theorem 5 following is a precise expression of the well known association between contact transformations and Legendre transformations. The reader may compare our approach with that of Burke (1985, §20), for example, who says that "the most common contact transformations are the Legendre transformations", notwithstanding the difference in dimensionality between the two. Arnold (1989, Appendix 4K) calls (45) a Legendre involution.

<u>Theorem 5</u>

The particular lift transformation (45) maps the lift of a function $y(x)$ onto the lift of its Legendre dual function $Y(x)$.

Proof

First we note that the lift of a pole $x = a$, $y = b$ with constant a and b is any vertical line above it with a finite range of z, and (45) maps this into the horizontal line $Y = aX - b$, $Z = a$, which is the lift of the polar of the original pole.

When the pole x, y moves on a smooth nonlinear plane curve $y = y(x)$ with slope $X = dy/dx$, the polar (46) envelops the smooth plane curve $Y = Y(X)$ with slope $x = dY/dX$, by the standard envelope properties of a Legendre transformation (e.g. see Theorem 2.3 of Sewell, 1987). The Legendre transformation now maps each plane curve into the other. Neither need be convex, nor even single valued. By adding $(45)_1$ and $(45)_3$ to these Legendre gradient properties we obtain

$$z = \frac{dy}{dx} \quad \text{and} \quad Z = \frac{dY}{dX}. \tag{47}$$

□

The convention described just before (8) allows us to say that a straight line $y = m(x - a) + b$ with fixed m is in contact with the point a, b on it. Their lifts are vertical and horizontal lines intersecting at $x = a$, $y = b$, $z = m$. Then (45) is a contact transformation which maps these lifts into the horizontal line $Y = aX - b$, $Z = a$ and the vertical line above $X = m$, $Y = ma - b$, which also intersect, thus illustrating transformation of this type of contact. In accordance with Theorem 5, $Y = aX - b$ is the polar of the pole a, b, and the polar of $m, ma - b$ is $y = m(x - a) + b$.

Any two curves in the x,y plane which are in contact will be converted by any lift transformation, such as all those in (20)–(22), and (43) and (45) in particular, into another pair of plane curves in the X,Y plane which are in contact, so that we may now call the lift transformations contact transformations. Theorem 5 shows that (45) has the additional feature that the pair of curves in the X,Y plane are the Legendre transforms of the pair of respective curves in the x,y plane.

The following example includes an explicit illustration of a pair of plane curves in contact. The family of regular lifted curves (8) parametrized by u for each fixed m can also be written

$$y = mx - \tfrac{1}{2}(1 + m^2)x^2 , \quad z = m - (1 + m^2)x . \tag{48}$$

The envelope of the plane curves $(48)_1$ for varying m has two disjoint parts, consisting of their common intersection point at $x = y = 0$ whose lift is any finite vertical line above it, and a parabola whose regular lift is

$$y = \tfrac{1}{2}(1 - x^2) , \quad z = - x . \tag{49}$$

Intersection of (48) with the vertical singularity takes place at

$$x = y = 0 , \quad z = m . \tag{50}$$

This represents the type of contact, defined in §2, whereby $x = y = 0$ lies on $(48)_1$ for each m. The intersection of (48) and (49) is, provided $m \neq 0$, at

$$x = \frac{1}{m} , \quad y = \tfrac{1}{2}\left(1 - \frac{1}{m^2}\right) , \quad z = - \frac{1}{m} , \tag{51}$$

so $(48)_1$ touches $(49)_1$ at $(51)_{1,2}$. Equation $(48)_1$ is the trajectory of a particle projected with initial slope m in vacuo over a flat Earth, with distances y measured vertically upwards and x horizontally. The initial speed U of projection is regarded as assigned. The parameter $u = Ut$ in (8), where t is time. For simplicity we have chosen units such that the acceleration due to gravity is $g = U^2$. Equation $(49)_1$ is the parabola of safety from all such trajectories having fixed U but any m. Equation $(51)_3$ proves that every trajectory touches the parabola of safety where the local velocity vector has turned through

90° from its initial direction (a pleasing little theorem in itself), as shown in Fig. 1.

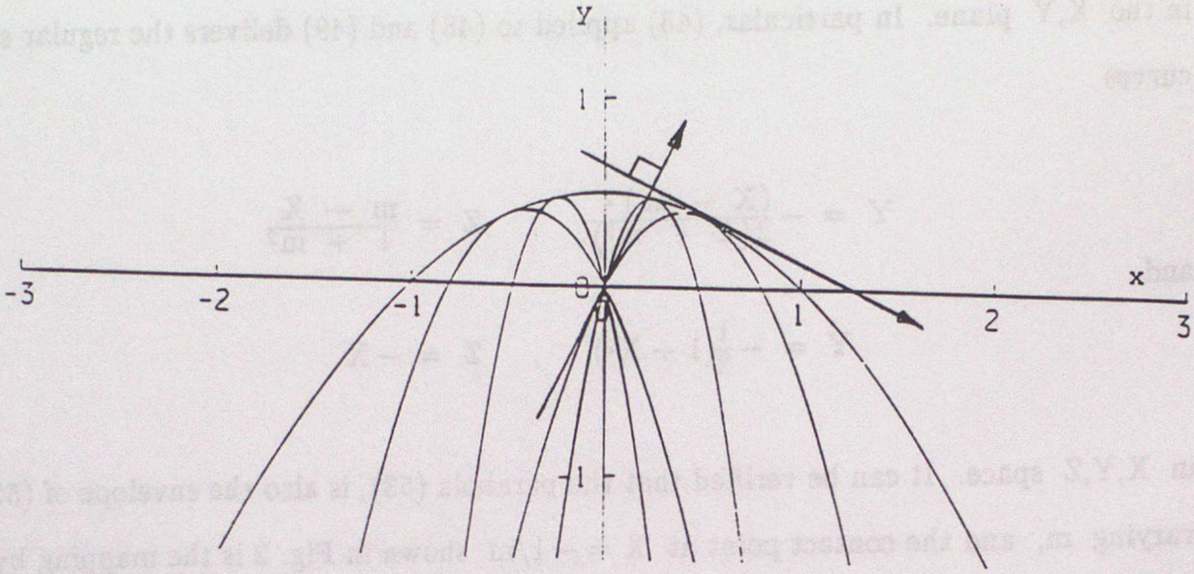

Fig. 1 Trajectories touching the parabola of safety.

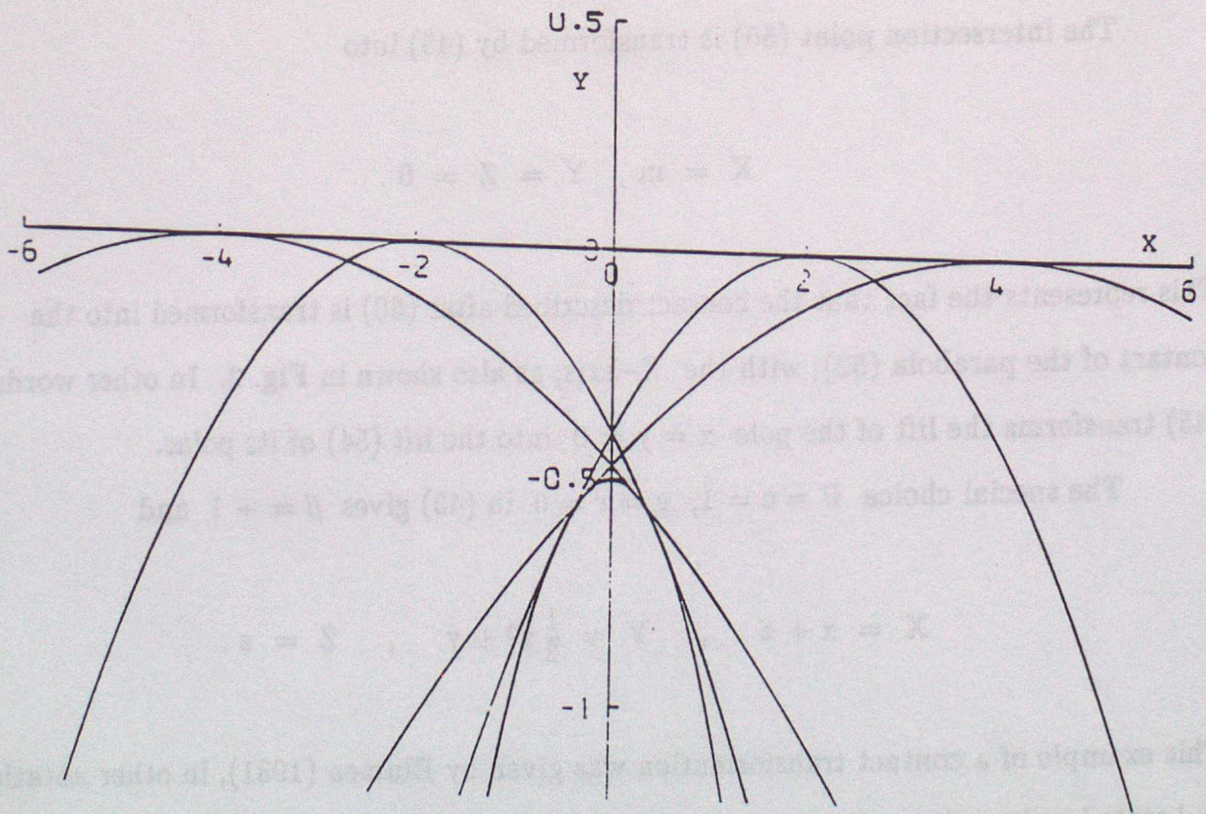

Fig. 2 Legendre duals of trajectory and envelope.

The reader can use any of the contact transformations (20)–(22) to illustrate how the plane contacting curves $(48)_1$ and $(49)_1$ transform into another pair of contacting curves in the X,Y plane. In particular, (45) applied to (48) and (49) delivers the regular space curves

$$Y = -\frac{(X - m)^2}{2(1 + m^2)} \quad , \quad Z = \frac{m - X}{1 + m^2} \tag{52}$$

and

$$Y = -\frac{1}{2}(1 + X^2) \quad , \quad Z = -X \tag{53}$$

in X,Y,Z space. It can be verified that the parabola $(53)_1$ is also the envelope of $(52)_1$ for varying m, and the contact point at $X = -1/m$ shown in Fig. 2 is the mapping by the contact transformation (45) of the original contact point $(51)_{1,2}$. It can also be verified that $(52)_1$ is the Legendre dual of $(48)_1$, thus illustrating Theorem 5 that the lift (48) is mapped by (45) into the lift of (52). The same applies to (49) and (53).

The intersection point (50) is transformed by (45) into

$$X = m\,, \quad Y = Z = 0\,. \tag{54}$$

This represents the fact that the contact described after (50) is transformed into the contact of the parabola $(52)_1$ with the X–axis, as also shown in Fig. 2. In other words, (45) transforms the lift of the pole $x = y = 0$ into the lift (54) of its polar.

The special choice $P = c = 1$, $g = r = 0$ in (43) gives $\beta = +1$ and

$$X = x + z \quad , \quad Y = \frac{1}{2} z^2 + y \quad , \quad Z = z\,. \tag{55}$$

This example of a contact transformation was given by Blumen (1981), in other notation, and stated to be a "one–dimensional analog" of the geostrophic coordinate transformation in meteorology. We shall explain and extend this statement in §9.

When we apply (55) to the regular lifts (48) and (49) we get, provided $m \neq 0$, the regular lift

$$Y = \frac{1}{2} - \frac{X}{m} + \frac{1}{2}\left[\frac{1}{m^2} + 1\right]X^2 \ , \quad Z = -\frac{1}{m} + \left[\frac{1}{m^2} + 1\right]X \tag{56}$$

and the vertical singular lift

$$X = 0 \ , \quad Y = \frac{1}{2} \ , \quad Z = -x \tag{57}$$

respectively. These intersect at the transform $0, 1/2, -1/m$ of (51), and the transformed contact is of the type that the point $X = 0$, $Y = 1/2$ lies on the parabola $(56)_1$. This illustrates that a lift transformation sometimes does not transform a regular lift into another regular lift. Moreover (55) applied to the vertical singular lift of the intersection $x = y = 0$ of $(48)_1$ delivers the regular lift of $Y = \frac{1}{2}X^2$. The envelope of $(56)_1$ for varying m consists of this parabola, and the common intersection point $X = 0$, $Y = \frac{1}{2}$, as shown in Fig. 3.

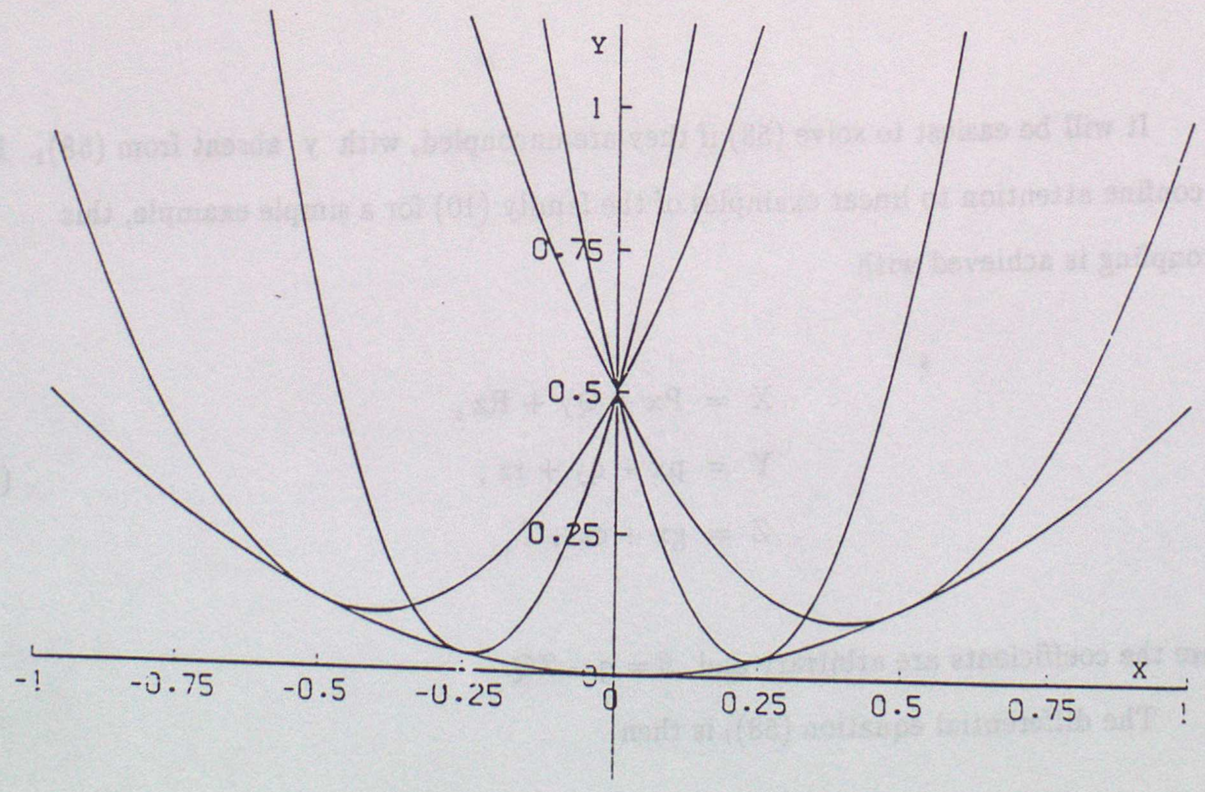

Fig. 3 Parabolas $(56)_1$ and their envelope.

6. Families of special lift transformations

In this Section we give up the requirement that (16) holds. We illustrate briefly how some lifts can still transform into lifts, even though Theorem 1 and its consequences are no longer available.

Theorem 6

If the transformation (10) is such that we can find a regular lifted curve satisfying the pair of differential equations

$$\gamma \frac{dz}{dx} + (\alpha + \beta z) = 0 , \quad \frac{dy}{dx} = z , \tag{58}$$

so that the curve depends on such (10), then the transformed curve is also a lift.

Proof

The result follows from (13) and (18) if $\beta \neq 0$ (and trivially so if $\beta = 0$).

□

It will be easiest to solve (58) if they are uncoupled, with y absent from $(58)_1$. If we confine attention to linear examples of the family (10) for a simple example, this uncoupling is achieved with

$$\begin{aligned} X &= Px + Qy + Rz , \\ Y &= px + qy + rz , \\ Z &= gx + cz + f , \end{aligned} \tag{59}$$

where the coefficients are arbitrary and $\beta = q - ZQ$.

The differential equation $(58)_1$ is then

$$[r + (gx + cz + f)R] \frac{dz}{dx} + p - fP - gPx + (q - cP - fQ)z - gQzx - cQz^2 = 0 \tag{60}$$

which is amenable to solution by elementary methods. For example, the equation is exact if

$$-gR = q - cP - fQ - Q(gx + 2cz)\,. \tag{61}$$

This is satisfied for all x and z if

$$Q = 0 \quad \text{and} \quad q = cP - gR\,. \tag{62}$$

Then $\beta = q$, and (60) is satisfied by any conic of the type

$$\frac{1}{2}gPx^2 + gRxz + \frac{1}{2}cRz^2 + (fP - p)x + (fR - r)z = \text{constant}. \tag{63}$$

As one of many possible illustrations we have the following particular simple consequence.

<u>Theorem 7</u>

The transformation

$$\begin{aligned} X &= \beta x + z\,, \\ Y &= [1 + (r - m)\beta]x + \beta y + rz\,, \\ Z &= z + r - m\,, \end{aligned} \tag{64}$$

with arbitrary constants $\beta \neq 0$, r and m, transforms the lifted curve

$$x = \frac{1}{2}z^2 - mz \quad , \quad y = \frac{1}{3}z^3 - \frac{1}{2}mz^2 \tag{65}$$

into another lifted curve in X, Y, Z space.

Proof

The result follows from (59) and (63) by choosing $g = 0$, $c = R = 1$, $f = r - m$, $p = 1 + (r - m)\beta$. Then $(65)_2$ follows from $(65)_1$ and $(58)_2$. □

The curve (65) is the same as (9), and it does have an isolated singularity at $z = m$, but is regular elsewhere. It is readily verified directly that the transformed curve in X, Y, Z space satisfies (13).

If (61) is not satisfied, then other methods are required. For example, when $g = 0$ (60) is a separable equation with a solution which may be non—algebraic.

7. Higher dimensions

There is one family of general lift transformations from R^{2n+1} to another R^{2n+1}, for any positive integer n, which can be briefly recorded here, because it is an immediate generalization of (43). The reader will easily be able to verify the following result, a special case of which is used in §9.

Theorem 8

Let P_{ij} and c_{ij} be typical components of any given symmetric $n \times n$ matrices, and let r_i be the typical component of any given $n \times 1$ matrix. Using the summation convention for repeated suffixes, the mappings

$$
\begin{aligned}
X_i &= P_{ij}\, x_j + z_i\,, \\
Y &= \frac{1}{2}(c_{pq} P_{qp} \pm 1) P_{ij}\, x_i\, x_j + \frac{1}{2} c_{ij}\, z_i\, z_j + (c_{pq} P_{qp} \pm 1) z_i\, x_i + r_i\, P_{ij}\, x_j \mp y + r_i\, z_i, \\
Z_i &= (c_{pq} P_{qp} \pm 1) x_i + c_{ij}\, z_j + r_i
\end{aligned}
\tag{66}
$$

are general lift transformations in the sense that first differentials satisfy

$$
dY - Z_i\, dX_i = \mp (dy - z_i\, dx_i)\,. \tag{67}
$$

8. Generalized duality

Here we augment a viewpoint which is familiar from, for example, the text of Morse and Feshbach (1953, p. 288), and we indicate how it is related to the approach of §§2–7. Let any given function $S(x, y, X, Y)$ of four variables be used to define an equation

$$S(x, y, X, Y) = 0 . \tag{68}$$

For each fixed x, y point, (68) is the implicit description of a curve C (say) in X, Y space; and for each fixed X, Y point (68) is the implicit description of a curve c (say) in x, y space. If the point x, y is now moved along another curve λ (say), the varying curves C will envelop another curve E (say), their envelope. We may call λ and E generalized dual curves. If the point X, Y is moved along another curve Λ (say), the varying curves c will envelop another curve e (say), their envelope. We may call Λ and e generalized dual curves also. This construction provides a pair of generalized dual mappings

$$\lambda \longrightarrow E \quad \text{and} \quad \Lambda \longrightarrow e , \tag{69}$$

each of one plane curve into another. It is worth recalling that envelopes do not have to be convex or minimizing curves, for example because of the presence on them of cusps or inflexions. In applications envelopes may be wave fronts (e.g., see Arnold 1989, 1990), for example. Fixed points may be disjoint parts of some envelopes, as we have seen.

We can define a line element of λ to be a point on λ together with the tangent line to λ at that point. The construction of a differentiable envelope E is such that the mapping $\lambda \longrightarrow E$ can also be regarded as a mapping of each line element of λ into a line element of E; and $\Lambda \longrightarrow e$ does the same for the line elements of Λ and e. When the functions in (1) are analytic, a line element of λ is specified by the two coordinates x, y and the value z of (6). This illustrates that the definition of line element has less ambiguity at singular points than the definition of lift in (3) (the definitions are the same at regular points). Notwithstanding the differences between these definitions, the verbal argument above for the mapping of line elements suggests that there will be circumstances in which the lift of λ maps into the lift of E (and that the lift of Λ maps into the lift of e). To identify such circumstances precisely, as illustrated in Theorem 10, we require the following preliminary result.

Theorem 9

When λ has the form (1), and E has the form $X = X(v), Y = Y(v)$ where v is another parameter related to u by a function $u(v)$, (68) has the properties

$$\frac{\partial S}{\partial x}\frac{dx}{du} + \frac{\partial S}{\partial y}\frac{dy}{du} = 0 \quad \text{and} \quad \frac{\partial S}{\partial X}\frac{dX}{dv} + \frac{\partial S}{\partial Y}\frac{dY}{dv} = 0 \,. \tag{70}$$

Proof

By definition, E is the relation between X and Y which is obtained by eliminating u between $S(x(u), y(u), X, Y) = 0$ and $dS/du = 0$. Then $(70)_1$ is immediate, and $(70)_2$ follows from $dS/dv = 0$ applied to
$S(x(u(v)), y(u(v)), X(v), Y(v)) = 0.$ □

Particular illustrations of (68) are available whenever it is possible to eliminate z between the first two equations of (10), whatever the function $Z(x, y, z)$ may be. Sufficient conditions for this are $\partial X/\partial z \neq 0$ or $\partial Y/\partial z \neq 0$. For example, Legendre duality is recovered in the special case $S(x, y, X, Y) = xX - y - Y$ obtained by eliminating z from the first two of (45), and c and C are then straight lines called the polars of the respective fixed points (called poles). Because $S(x, y, X, Y) = S(X, Y, x, y)$, the two polars c and C are the same whenever the poles are the same. However, this type of symmetry is not present in other cases, such as the $S(x, y, X, Y) = \frac{1}{2}(x - X)^2 + y - Y$ implied by (55). These two examples are included in that provided by (43), namely

$$S(x, y, X, Y) = \frac{1}{2}(cP \pm 1)Px^2 + rPx \mp y$$
$$+ \frac{1}{2}c(Px - X)^2 - [(cP \pm 1)x + r](Px - X) - Y\,. \tag{71}$$

The Legendre case provides an example of Theorem 9 in which, at regular points, we can use $u = x$ and $v = X$, whence (70) becomes the familiar $X = dy/dx$ and $x = dY/dX$.

Theorem 10

Any transformation of the form

$$X = X(x, y, z)\;, \quad Y = Y(x, y, z)\;, \quad Z = \frac{\partial Y/\partial z}{\partial X/\partial z} \tag{72}$$

in which

$$\frac{\partial X}{\partial z} \neq 0 \tag{73}$$

has the following properties.

(i) There exists a function $z(x, y, X)$ inducing the definition

$$S(x, y, X, Y) = Y(x, y, z(x, y, X)) - Y \tag{74}$$

which provides an example of (68), and which has the gradients

$$\frac{\partial S}{\partial x} = \alpha\;, \quad \frac{\partial S}{\partial y} = \beta\;, \quad \frac{\partial S}{\partial X} = Z\;, \quad \frac{\partial S}{\partial Y} = -1\,. \tag{75}$$

(ii) Each fixed point $x = a, y = b$ is mapped by $(72)_1$ and $(72)_2$ into a plane curve C consisting entirely of regular points and expressible explicitly as

$$Y = Y(X) \quad \text{with gradient} \quad \frac{dY}{dX} = \frac{\partial Y/\partial z}{\partial X/\partial z}\,. \tag{76}$$

(iii) The whole transformation (72) maps the lift of the fixed point into the lift of C.

(iv) When the x, y point moves along a plane curve λ, and (72) is required to be a general lift transformation, z is the height of the lift of λ. Then C varies, and (72) maps the lift of λ into the lift of the envelope E of the curves C.

Proof

(i) Condition (73) is sufficient for $(72)_1$ to have an inverse function $z(x, y, X)$ whose gradients satisfy

$$0 = \frac{\partial X}{\partial x} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial x} \,, \quad 0 = \frac{\partial X}{\partial y} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial y} \,, \quad 1 = \frac{\partial X}{\partial z}\frac{\partial z}{\partial X} \,. \tag{77}$$

This inverse can be used in $(72)_2$ to construct the function (74) whose gradients are those stated in (75), by using (14), $(72)_3$ and (77).

(ii) Equations $(72)_1$ and $(72)_2$ map the fixed point into a curve C expressible in parametric form as $X = X(a, b, z)$, $Y = Y(a, b, z)$ with z as the parameter, and (73) is the definition that all points of C are regular. C is expressible alternatively from (68) and (74) as $S(a, b, X(a, b, z), Y(a, b, z)) = 0$ or $Y = Y(a, b, z(a, b, X))$. The chain rule with $(77)_3$ delivers $(76)_2$.

(iii) The lift of a, b is the vertical singularity above it, which is parametrized by z. This lift is mapped by the whole of (72) into a space curve

$$Y = Y(X) \quad \text{with height} \quad Z(X) = \frac{dY}{dX} \,, \tag{78}$$

by equating $(72)_3$ and $(76)_2$. But (78) is the definition of the lift of C when, as proved in (ii), all the points of C are regular. It will be noticed that this proof only needs the last part of Theorem 2, and it does not use the conditions $\alpha + \beta z = 0$ in $(16)_1$, $\beta \neq 0$ in (19), or the further hypothesis used to get $(17)_1$.

(iv) The gradient of λ satisfies $(70)_1$ in general, and since (75) applies in the case of (72) with (73),

$$\alpha \frac{dx}{du} + \beta \frac{dy}{du} = 0 . \tag{79}$$

When (72) is a general lift transformation Theorem 2 tells us that $\alpha + \beta z = 0$ and $\beta \neq 0$, and then (79) implies

$$\frac{dy}{du} = z \frac{dx}{du} \tag{80}$$

so that z is the height of the lift of λ.

The envelope E of C has a slope which satisfies $(70)_2$, and therefore

$$\frac{dY}{dv} = Z \frac{dX}{dv} \tag{81}$$

from (75). By comparison with (13) (with u there now replaced by v), we deduce that Z is the height of the lift of E. □

At regular points of λ we can choose $u = x$, and at regular points of E we can choose $v = X$. The proof of Theorem 10 for the Legendre case is given explicitly in Theorem 5.

9. A meteorological application

In this Section we use a notation associated with the physics of the problem, in place of the neutral notation of the previous mathematics.

Certain mid–latitude motions of the atmosphere on a sub–continental scale, including allowance for the presence of fronts, can be analyzed using a cartesian system of spatial coordinates $\underline{x}$, z and time t as independent variables. Here $\underline{x}$ is the true horizontal position vector, and z is a pressure–related pseudo–height. The geostrophic coordinate transformation is a mapping to an alternative system of independent variables called geostrophic coordinates $\underline{M}$, z, t, where $\underline{M}$ is the horizontal momentum vector. The active part of the mapping, between $\underline{x}$ and $\underline{M}$, can be expressed, in the notation of Chynoweth and Sewell (1989, 1991), as a Legendre transformation with properties

$$\underline{M} = \frac{\partial P}{\partial \underline{x}} \quad , \quad \underline{x} = \frac{\partial S}{\partial \underline{M}} \quad , \quad \underline{M}.\underline{x} = P + S \tag{82}$$

where $P(\underline{x})$ and $S(\underline{M})$ are two scalar functions (each of z and t also). This use of S is quite different from that in (68). The modified geopotential (Cullen and Purser, 1984) is $P(\underline{x}) = \phi(\underline{x}) + \frac{1}{2}\underline{x}^2$ in terms of the true geopotential function $\phi(\underline{x})$. If we introduce a new two dimensional vector variable $\underline{p}$, we can extend Blumen's (1981) result concerning (55) to higher dimensions as follows.

<u>Theorem 11</u>

The transformation

$$\underline{M} = \underline{x} + \underline{p} \quad , \quad Y = \frac{1}{2}\underline{p}^2 + y \quad , \quad \underline{Z} = \underline{p} \tag{83}$$

from $\underline{x}$, y, $\underline{p}$ space to $\underline{M}$, Y, $\underline{Z}$ space is a higher dimensional general lift transformation from one R^5 to another, in the sense of (67) for $n = 2$ there.

In particular it maps the lift

$$y = \phi(\underline{x}) \quad , \quad \underline{p} = \frac{\partial \phi}{\partial \underline{x}} \tag{84}$$

of the geopotential into the lift

$$Y = \frac{1}{2}\underline{M}^2 - S(\underline{M}) \quad , \quad \underline{Z} = \underline{M} - \underline{x}\,. \tag{85}$$

Proof

The first differentials of (83) satisfy

$$dY - \underline{Z}.d\underline{M} = \underline{p}.d\underline{p} + dy - \underline{p}.(d\underline{x} + d\underline{p}) = dy - \underline{p}.d\underline{x}\,, \tag{86}$$

which is a particular example of (67), and we can say that (83) is a higher dimensional general lift transformation belonging to the family (66). In this case, in fact,

$$\beta = \frac{\partial Y}{\partial y} - \underline{Z}.\frac{\partial \underline{M}}{\partial y} = 1\,. \tag{87}$$

When (84), which makes the right side of (86) vanish, is inserted into (83), and $(82)_1$ is used to express $\underline{x}$ in terms of $\underline{M}$, we obtain (85), and the left side of (86) vanishes.

□

Purser (1993) has carried further the applications of contact transformations, based upon the viewpoint of (68), to a wide class of semigeostrophic theories, including vortex dynamics.

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