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Some notes on primitive and quasi-equilibrium equations with a hybrid vertical coordinate and remarks on Hamiltonian structure

by

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23rd March 1993

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1 Preliminaries

1.1 *An outline of transformation theory*

In this paper (§(2) - §(4)) we will be concerned with the use of hybrid vertical coordinates in non-hydrostatic, quasi-hydrostatic and quasi-equilibrium models. We shall also remark (in §(5)) on the work of Holm & Long (1989) and Shepherd (1990), who have formulated Hamiltonian versions of the hydrostatic and non-hydrostatic primitive equations respectively.

Before we consider a particular hybrid coordinate system we shall review the theory of coordinate transformations and remark on the use of the chain rule and tensor methods in these models. Kasahara (1974) gave a comprehensive review of the coordinate transformations for various vertical coordinate systems used for the hydrostatic primitive equations, and much of that work can be used for our purposes here. However, we need to make some revisions in the case of the non-hydrostatic equations, to be discussed in the next section, and it is appropriate to comment on the relative merits and on the outcome of using generalized tensor methods (c.f. Dutton (1976), Pielke & Martin (1981)) or the chain rule.

If we wish to retain all the properties of a set of equations under a general coordinate transformation then it is, without doubt, sensible to write them out in a generally covariant way. This means that the equations of motion, written in terms of tensors, are valid in any coordinate system. However, we frequently consider approximations to the fully three dimensional version of Newtonian hydrodynamics and in these cases we must exercise care when applying transformation theory. If we approximate the Navier-Stokes equations in any way, then we must decide whether to make this approximation *before or after* making a coordinate transformation. To make this point clear, consider the following example: Suppose we write out the Navier-Stokes equations in spherical polar coordinates and make the assumption of hydrostatic balance. Then if we wish to transform this set of hydrostatic equations to another coordinate system (most possibly a non-orthogonal one) then we may *either* perform a series of chain-rule calculations on the approximate equations, assuming that the new coordinates are appropriately well-defined functions of the old coordinates, *or* we may start with a covariant form of the full Navier-Stokes equations, calculate the explicit tensorial relationships (e.g. metric and connection coefficients), transform the equations and *then* impose the assumption of hydrostatic balance. In general, these two methods will give different results. (For instance, the fully covariant method involves transformation of the rotation or coriolis vector so that whereas the original vector is comprised of two components, the transformed vector will have three components. Sometimes one gets some remarkable cancellations between the new terms, however, and this is a crucial point for the hydrostatic equations, the vertical momentum equation will often be considerably more complicated (see Dutton (1976) or James (1991) for an example.)) Of course, we should point out that the two sets of equations being considered here, predict the evolution of two different sets of velocities. That is, the chain rule will yield equations for the velocities with respect to the original set of coordinates, say (u, v, w), whilst it is implicit in the definition of a covariant method that the components of the velocity vector must be with respect to the new coordinate system. Thus if one were to consider, for example, isentropic coordinates, then the velocity vector would have components along the isentropic surfaces.

1.2 *A review of basic techniques*

For the purposes of this paper we shall work exclusively with the chain rule for the following reasons. First, it is desirable to write the momentum equations in terms of the local eulerian velocities (u, v, w), rather than velocities relative to the hybrid coordinates. It is the latter that result in complicated vertical momentum equations and, as we shall demonstrate, we may diagnose the vertical velocity relative to the new coordinates. Second, when working with the non-hydrostatic equations, we avoid any necessity to refer to approximations that would lead to significantly different equations, that depend on the procedure carried out when 'transforming' the approximation to various coordinate systems.

We shall now review some of the basic rules of transformation theory. Let the independent variables of a geometric or physical height-based coordinate system be (λ, ϕ, r, t). Here $r = z + a$, where a is the mean radius of the earth. We will denote the generalized coordinates by (λ, ϕ, η, t), with

$$\eta = \eta(\lambda, \phi, r, t) \ .$$

Assuming a single-valued, monotonic relationship between r and η, it follows that

$$r = r(\lambda, \phi, \eta, t) \ .$$

The following relationships between partial derivatives exist: Let A be any scalar function which may be expressed in either of two ways depending upon whether r or η is chosen as the vertical coordinate

$$\left.\frac{\partial A}{\partial c}\right|_r = \left.\frac{\partial A}{\partial c}\right|_\eta - \left.\frac{\partial A}{\partial r}\frac{\partial r}{\partial c}\right|_\eta \quad , \tag{1}$$

where, throughout this paper, $c = \lambda, \phi$ or t. We also have

$$\frac{\partial A}{\partial r} = \frac{\partial A}{\partial \eta}\frac{\partial \eta}{\partial r} \quad . \tag{2}$$

The material derivative is written in the generalized system as follows

$$\frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right|_\eta + \mathbf{v}.\nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta} \quad , \tag{3}$$

where $\dot{\eta}$ is the generalized vertical velocity $\frac{D\eta}{Dt}$, and $\mathbf{v} \leftrightarrow (u, v, 0)$. $\hat{w}$ $(= \left.\frac{Dr}{Dt}\right|_\eta)$ and $\dot{\eta}$ are related as follows

$$\dot{\eta} = \frac{\partial \eta}{\partial r}\left(\hat{w} - \left.\frac{\partial r}{\partial t}\right|_\eta - \mathbf{v}.\nabla_\eta r\right) \quad . \tag{4}$$

The continuity equation in the generalized system may be obtained as follows. The local eulerian vertical velocity is given by (c.f. (4))

$$\hat{w} = \left.\frac{\partial r}{\partial t}\right|_\eta + \mathbf{v}.\nabla_\eta r + \dot{\eta}\frac{\partial r}{\partial \eta} \quad , \tag{5}$$

and by the chain rule

$$\frac{\partial \hat{w}}{\partial r} = \frac{\partial \hat{w}}{\partial \eta}\frac{\partial \eta}{\partial r} \quad ,$$

and thus, from (5)

$$\begin{aligned} \frac{\partial \hat{w}}{\partial r} &= \frac{\partial \eta}{\partial r}\left(\frac{\partial}{\partial t}\left(\frac{\partial r}{\partial \eta}\right) + \mathbf{v}.\nabla_\eta\left(\frac{\partial r}{\partial \eta}\right) + \frac{\partial \mathbf{v}}{\partial \eta}.\nabla_\eta r + \dot{\eta}\frac{\partial}{\partial \eta}\left(\frac{\partial r}{\partial \eta}\right)\right) + \frac{\partial \dot{\eta}}{\partial \eta} \\ &= \frac{\partial \eta}{\partial r}\left(\frac{D}{Dt}\left(\frac{\partial r}{\partial \eta}\right) + \frac{\partial \mathbf{v}}{\partial \eta}.\nabla_\eta r\right) + \frac{\partial \dot{\eta}}{\partial \eta} \quad . \end{aligned} \tag{6}$$

From (1) and (2)

$$\nabla_r.\mathbf{v} = \nabla_\eta.\mathbf{v} - \frac{\partial \eta}{\partial r}\nabla_\eta r.\frac{\partial \mathbf{v}}{\partial \eta} \quad , \tag{7}$$

and thus if we write the continuity equation (13) as

$$\frac{D\rho}{Dt} + \rho\nabla_r.\mathbf{v} + \rho\frac{\partial w}{\partial r} = 0 \tag{8}$$

we may re-write it for generalized coordinates, using (6) and (7), as

$$\begin{aligned}\frac{D\rho}{Dt} \quad &+ \quad \rho\left(\nabla_\eta.\mathbf{v} - \frac{\partial\eta}{\partial r}(\nabla_\eta r).\frac{\partial\mathbf{v}}{\partial\eta}\right) \\ &+ \quad \rho\left(\frac{\partial\dot{\eta}}{\partial\eta} + \frac{\partial\eta}{\partial r}\left[\frac{D}{Dt}\left(\frac{\partial r}{\partial\eta}\right) + \frac{\partial\mathbf{v}}{\partial\eta}.\nabla_\eta r\right]\right) = 0 \ .\end{aligned} \tag{9}$$

This simplifies to

$$\frac{D}{Dt}\log\left(\rho\frac{\partial r}{\partial\eta}\right) + \nabla_\eta.\mathbf{v} + \frac{\partial\dot{\eta}}{\partial\eta} = 0 \ , \tag{10}$$

or

$$\frac{\partial}{\partial t}\left[\rho\frac{\partial r}{\partial\eta}\right]_\eta + \nabla_\eta.\left(\rho\mathbf{v}\frac{\partial r}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\rho\dot{\eta}\frac{\partial r}{\partial\eta}\right) = 0 \ . \tag{11}$$

2 The non-hydrostatic equations

The three components of the Navier-Stokes equations written in spherical polar coordinates are

$$\left.\begin{aligned}\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} &= 0 \\ \frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} &= 0 \\ \frac{Dw}{Dt} - 2\Omega u\cos\phi - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0\end{aligned}\right\} \ . \tag{12}$$

The continuity and thermodynamic equations are

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} \;=\; 0 \tag{13}$$

$$\frac{D\theta}{Dt} \;=\; \left(\frac{\theta}{TC_p}\right)Q \ , \tag{14}$$

where Q is the diabatic heating rate. Perfect gas behaviour is assumed $p = \rho RT$.

We will consider a terrain-following hybrid coordinate based on height, of which the general form is given by

$$\eta = s \left[\frac{r - E(\lambda,\ \phi)}{s - E(\lambda,\ \phi)} \right] \quad , \tag{15}$$

where s is usually defined as a constant (generally defined as the top of the model), while $E(\lambda,\ \phi)$ is the terrain height. From this definition the following useful results may be obtained

$$\dot{\eta} \equiv \frac{D\eta}{Dt} = \frac{s}{(s-E)^2} \left((s-E)(\hat{w} - \dot{E}) + (r-E)\dot{E} \right) \quad , \tag{16}$$

where $\hat{w} \equiv \left. \frac{Dr}{Dt} \right|_{\eta}$, and

$$\frac{\partial \eta}{\partial r} = \frac{s}{s-E} \quad . \tag{17}$$

The momentum equations with all derivatives evaluated on η-surfaces are

$$\begin{aligned} \frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi} \right) (v \sin\phi - \hat{w}\cos\phi) & \\ + \quad \frac{1}{\rho r \cos\phi} \left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial r}\frac{\partial r}{\partial \lambda} \right) &= 0 \quad , \end{aligned} \tag{18}$$

$$\frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi} \right) u \sin\phi + \frac{v\hat{w}}{r} + \frac{1}{\rho r} \left(\frac{\partial p}{\partial \phi} - \frac{\partial p}{\partial r}\frac{\partial r}{\partial \phi} \right) = 0 \quad , \tag{19}$$

$$\frac{Dw}{Dt} - 2\Omega u \cos\phi - \frac{u^2 + v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \quad . \tag{20}$$

The continuity equation, using (11) and the (inverse of) (17), is

$$\frac{\partial}{\partial t} \left(\rho \frac{s-E}{s} \right) + \nabla_{\eta} \cdot \left(\rho \mathbf{v} \frac{s-E}{s} \right) + \frac{\partial}{\partial \eta} \left(\rho \dot{\eta} \frac{s-E}{s} \right) = 0 \quad . \tag{21}$$

The thermodynamic equation (14) is essentially in the appropriate form because it is written in terms of the material derivative. For further discussion of non-hydrostatic models, refer to White & Bromley (1990).

3 The quasi-hydrostatic equations in η-coordinates

3.1 *Transformation of a height coordinate model*

The three components of the quasi-hydrostatic equations in spherical geometry may be written

$$\left.\begin{aligned} \frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} &= 0 \\ \frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\phi} &= 0 \\ -2\Omega u\cos\phi - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0 \end{aligned}\right\} . \tag{22}$$

The terms on the right represent the frictional force per unit mass. The material derivative is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r} \ ,$$

for further explanation of notation see White & Bromley, (1990). The continuity and thermodynamic equations are the same as (13) and (14).

Using the transformation theory of §(1) we may write the momentum equations (22) in the generalized coordinates as follows:

$$\left.\begin{aligned} \frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - \hat{w}\cos\phi) + \frac{1}{\rho r\cos\phi}\left(\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial r}\frac{\partial r}{\partial\lambda}\right) &= 0 \\ \frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{v\hat{w}}{r} + \frac{1}{\rho r}\left(\frac{\partial p}{\partial\phi} - \frac{\partial p}{\partial r}\frac{\partial r}{\partial\phi}\right) &= 0 \\ -\left(2\Omega + \frac{u}{r\cos\phi}\right)u\cos\phi - \frac{v^2}{r} + g + \frac{1}{\rho}\frac{\partial p}{\partial\eta}\frac{\partial\eta}{\partial r} &= 0 \end{aligned}\right\} . \tag{23}$$

We note from (23)$_3$ that we have a modified quasi-hydrostatic relationship that we will denote as follows

$$\left.\frac{\partial p}{\partial r}\right|_{\eta} = -\rho g + \rho\left(2\Omega u\cos\phi + \frac{u^2+v^2}{r}\right) \equiv \mu\rho \ , \tag{24}$$

and thus we may re-write (23) substituting for the above

$$\left.\begin{array}{rcl}\dfrac{Du}{Dt}-\left(2\Omega+\dfrac{u}{r\cos\phi}\right)(v\sin\phi-\hat{w}\cos\phi)+\dfrac{1}{\rho r\cos\phi}\left(\dfrac{\partial p}{\partial\lambda}-\rho\mu\dfrac{\partial r}{\partial\lambda}\right) & = & 0\\ \dfrac{Dv}{Dt}+\left(2\Omega+\dfrac{u}{r\cos\phi}\right)u\sin\phi+\dfrac{v\hat{w}}{r}+\dfrac{1}{\rho r}\left(\dfrac{\partial p}{\partial\phi}-\rho\mu\dfrac{\partial r}{\partial\phi}\right) & = & 0\\ \dfrac{\partial p}{\partial r} & = & \rho\mu\end{array}\right\} . \tag{25}$$

In (23) and (25) the material derivative is given by (3).

Thus, by (25)$_3$, (11) becomes

$$\frac{\partial}{\partial t}\left[\frac{1}{\mu}\frac{\partial p}{\partial\eta}\right]_\eta+\nabla_\eta.\left(\frac{\mathbf{v}}{\mu}\frac{\partial p}{\partial\eta}\right)+\frac{\partial}{\partial\eta}\left(\frac{\dot{\eta}}{\mu}\frac{\partial p}{\partial\eta}\right)=0 \ . \tag{26}$$

3.2 *Transformation of a pressure coordinate model*

In this subsection we are interested in the version of the quasi-hydrostatic equations used in the unified model. When transforming from height coordinates $(\lambda,\ \phi,\ r)$ to pressure coordinates $(\lambda,\ \phi,\ p)$, we find that the pressure gradient terms where the derivatives are taken with respect to the angular variables, may be expressed as

$$\begin{array}{rcl}\left.\dfrac{\partial p}{\partial c}\right|_r & = & \left.\dfrac{\partial p}{\partial c}\right|_p-\dfrac{\partial p}{\partial r}\left.\dfrac{\partial r}{\partial c}\right|_p\\ & = & 0-\dfrac{\partial p}{\partial r}\left.\dfrac{\partial r}{\partial c}\right|_p \ .\end{array} \tag{27}$$

If the hydrostatic relationship is written in the form

$$\frac{\partial p}{\partial r}=\tilde{\mu}\rho \ ,$$

then

$$\left.\frac{\partial p}{\partial c}\right|_r=\rho\left.\frac{\partial\Phi}{\partial c}\right|_p \ ,$$

where $\Phi = gz$ is the geopotential and $\tilde{\mu}$ is equal to $-g$. We may write the (u, v) momentum equations, with $\tilde{w} \equiv \left.\frac{Dr}{Dt}\right|_p$, in the form

$$\left.\begin{aligned} \frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - \tilde{w}\cos\phi) + \frac{1}{\rho r\cos\phi}\frac{\partial\Phi}{\partial\lambda} &= 0 \\ \frac{Dv}{Dt} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{v\tilde{w}}{r} + \frac{1}{\rho r}\frac{\partial\Phi}{\partial\phi} &= 0 \end{aligned}\right\} , \qquad (28)$$

where the material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}.\nabla_p + \omega\frac{\partial}{\partial p} \ ,$$

and $\omega \equiv \frac{Dp}{Dt}$. Observe that the hydrostatic equation may be written

$$\left(-2\Omega u\cos\phi - \frac{u^2+v^2}{r} + g\right)\rho\frac{\partial r}{\partial p} + 1 = 0 \ ,$$

and thus multiplying by $\frac{\partial\Phi}{\partial r}$ gives

$$-\frac{\partial\Phi}{\partial p}\mu + \frac{gRT}{p} = 0 \ , \qquad (29)$$

where μ is defined by (24). It turns out that in order to get the expression for hydrostatic balance that is used in the model we must follow White & Bromley (1990) and make the approximation that

$$\left(-2\Omega u\cos\phi - \frac{u^2+v^2}{r}\right)\frac{\partial\Phi}{\partial p} = \frac{RT_s}{p}\left(-2\Omega u\cos\phi - \frac{u^2+v^2}{r}\right) \ , \qquad (30)$$

where T_s represents a horizontally averaged, hydrostatically balanced temperature profile. With this approximation (29) becomes

$$\frac{\partial\Phi}{\partial p} + \frac{RT_s}{gp}\left(-2\Omega u\cos\phi - \frac{u^2+v^2}{r}\right) + \frac{RT}{p} = 0 \ . \qquad (31)$$

The continuity equation is written

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial r}{\partial p}\frac{\partial p}{\partial\eta}\right) + \nabla_\eta.\left(\rho\mathbf{v}\frac{\partial r}{\partial p}\frac{\partial p}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\rho\dot{\eta}\frac{\partial r}{\partial p}\frac{\partial p}{\partial\eta}\right) = 0 \ , \qquad (32)$$

and thus using (24) we get

$$\frac{\partial}{\partial t}\left(\frac{1}{\mu}\frac{\partial p}{\partial \eta}\right) + \nabla_\eta . \left(\mathbf{v}\frac{1}{\mu}\frac{\partial p}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\dot{\eta}}{\mu}\frac{\partial p}{\partial \eta}\right) = 0 \ . \tag{33}$$

We should note that the transformation of the pressure coordinate model introduces *two* hydrostatic assumptions: On the one hand we need the *primitive equation* hydrostatic balance to write down the momentum equations (28) in pressure coordinates and then we arrive at another version of hydrostatic balance, so-called *quasi-hydrostatic* balance in (29). It is essentially this latter form that is used in the (full) continuity equation (33), however we find that the primitive equation hydrostatic balance is used in the forecast model equations. This amounts to selectively neglecting certain small (in the sense of scale analysis) terms in the hydrostatic balance, depending upon where in the equations they occur. The reason behind this procedure is that one can demonstrate that the resulting inertial, metric and coriolis terms in the equations (28) and (31) are isomorphic to the corresponding terms in the height coordinate version (22). The pressure coordinate version also retains the energy, potential vorticity and axial angular momentum principles. For further details of these matters see White & Bromley (1990).

4 The quasi-equilibrium model in η-coordinates

4.1 *The equations of motion and the transformation of a height-based model*

The quasi-equilibrium equations in spherical polar coordinates are usually written in pressure coordinates as

$$\left.\begin{array}{rcl} \dfrac{Du_g}{Dt} - 2\Omega v \sin\phi - \dfrac{uv_g}{a}\tan\phi + \dfrac{1}{a\cos\phi}\dfrac{\partial\Phi}{\partial\lambda} & = & 0 \\ \dfrac{Dv_g}{Dt} + 2\Omega u \sin\phi + \dfrac{uu_g}{a}\tan\phi + \dfrac{1}{a}\dfrac{\partial\Phi}{\partial\phi} & = & 0 \\ \dfrac{\partial\Phi}{\partial p} & = & -\dfrac{1}{\rho} \end{array}\right\} . \tag{34}$$

The material derivative in (34) is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{r}{a}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial p} .$$

The potential temperature is conserved in this model

$$\frac{D\theta}{Dt} = 0 . \tag{35}$$

In order to study integration schemes that may be used for both the non-hydrostatic equations and the quasi-equilibrium equations it will be convenient to write (34) in terms of a height based coordinate. To do this we note the result

$$\begin{aligned} \left.\frac{\partial\Phi}{\partial c}\right|_p &= \left.\frac{\partial\Phi}{\partial c}\right|_r - \left.\frac{\partial\Phi}{\partial p}\frac{\partial p}{\partial c}\right|_r \\ &= 0 - \frac{\partial\Phi}{\partial p}\frac{\partial p}{\partial c} , \end{aligned} \tag{36}$$

and use the hydrostatic relationship

$$\frac{\partial\Phi}{\partial p} = -\frac{1}{\rho} ,$$

in (36) to obtain

$$\left.\frac{\partial \Phi}{\partial c}\right|_p = \frac{1}{\rho}\left.\frac{\partial p}{\partial c}\right|_r \quad . \tag{37}$$

With this result we may write (34) in the form

$$\left.\begin{aligned} \frac{Du_g}{Dt} - 2\Omega v \sin\phi - \frac{uv_g}{a}\tan\phi + \frac{1}{a\rho\cos\phi}\frac{\partial p}{\partial \lambda} &= 0 \\ \frac{Dv_g}{Dt} + 2\Omega u \sin\phi + \frac{uu_g}{a}\tan\phi + \frac{1}{a\rho}\frac{\partial p}{\partial \phi} &= 0 \\ g + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0 \end{aligned}\right\} \quad . \tag{38}$$

Here, the material derivative is defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial \lambda} + \frac{r}{a}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial r} \quad .$$

To transform this set into hybrid coordinates with η given as in (15) we proceed as in §(1) and §(2) and obtain

$$\left.\begin{aligned} \frac{Du_g}{Dt} - 2\Omega v \sin\phi - \frac{uv_g}{a}\tan\phi + \frac{1}{a\rho\cos\phi}\left(\frac{\partial p}{\partial \lambda} + g\rho\frac{\partial r}{\partial \lambda}\right) &= 0 \\ \frac{Dv_g}{Dt} + 2\Omega u \sin\phi + \frac{uu_g}{a}\tan\phi + \frac{1}{a\rho}\left(\frac{\partial p}{\partial \phi} + g\rho\frac{\partial r}{\partial \phi}\right) &= 0 \\ g + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0 \end{aligned}\right\} \quad , \tag{39}$$

with the understanding that the derivatives with respect to t, λ and ϕ are evaluated at constant η, and that the material derivative is now given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial \lambda} + \frac{r}{a}\frac{\partial}{\partial \phi} + \dot{\eta}\frac{\partial}{\partial \eta} \quad , \tag{40}$$

The continuity equation is given by (21).

4.2 *Transformation of the pressure coordinate model*

Using the same hybrid vertical coordinate η, as in the previous subsection, the equations (34) with a suitable representation of the hydrostatic balance, become

$$\left.\begin{array}{rcl} \dfrac{Du_g}{Dt} - 2\Omega v \sin\phi - \dfrac{uv_g}{a}\tan\phi + \dfrac{1}{a\cos\phi}\left(\dfrac{\partial\Phi}{\partial\lambda} - \dfrac{\partial\Phi}{\partial r}\dfrac{\partial r}{\partial\lambda}\right) & = & 0 \\ \dfrac{Dv_g}{Dt} + 2\Omega u \sin\phi + \dfrac{uu_g}{a}\tan\phi + \dfrac{1}{a}\left(\dfrac{\partial\Phi}{\partial\phi} - \dfrac{\partial\Phi}{\partial r}\dfrac{\partial r}{\partial\phi}\right) & = & 0 \\ \dfrac{\partial\Phi}{\partial r} & = & \dfrac{g\theta}{\theta_0} \end{array}\right\} . \tag{41}$$

We may substitute the hydrostatic relationship into the momentum equations and obtain

$$\left.\begin{array}{rcl} \dfrac{Du_g}{Dt} - 2\Omega v \sin\phi - \dfrac{uv_g}{a}\tan\phi + \dfrac{1}{a\cos\phi}\left(\dfrac{\partial\Phi}{\partial\lambda} - \dfrac{g\theta}{\theta_0}\dfrac{\partial r}{\partial\lambda}\right) & = & 0 \\ \dfrac{Dv_g}{Dt} + 2\Omega u \sin\phi + \dfrac{uu_g}{a}\tan\phi + \dfrac{1}{a}\left(\dfrac{\partial\Phi}{\partial\phi} - \dfrac{g\theta}{\theta_0}\dfrac{\partial r}{\partial\phi}\right) & = & 0 \end{array}\right\} . \tag{42}$$

The thermodynamic equation remains unaltered as it is expressed entirely in terms of the invariant material derivative which is given by (40). The continuity equation, written in the form

$$\frac{\partial}{\partial t}\left[\rho\frac{\partial r}{\partial\eta}\right]_\eta + \nabla_\eta\cdot\left(\rho\mathbf{v}\frac{\partial r}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\rho\dot{\eta}\frac{\partial r}{\partial\eta}\right) = 0 \ , \tag{43}$$

becomes

$$\frac{\partial}{\partial t}\left[\rho\frac{\partial\Phi}{\partial\eta}\frac{\theta_0}{g\theta}\right]_\eta + \nabla_\eta\cdot\left(\rho\mathbf{v}\frac{\partial\Phi}{\partial\eta}\frac{\theta_0}{g\theta}\right) + \frac{\partial}{\partial\eta}\left(\rho\dot{\eta}\frac{\partial\Phi}{\partial\eta}\frac{\theta_0}{g\theta}\right) = 0 \ . \tag{44}$$

One may wish, however, to use the form of the continuity equation (21). The vertical velocity $\dot{\eta}$, may be diagnosed as in §(1.2) (16), where the eulerian vertical velocity is given by $\hat{w} \equiv \left.\dfrac{Dr}{Dt}\right|_\eta$.

4.3 *A version of the quasi-equilibrium equations retaining metric and coriolis terms*

The material covered in subsections §(3.1) and §(3.2) incorporated the shallow atmosphere approximation in the quasi-equilibrium model and the 'usual' metric and coriolis terms were neglected. In this subsection we shall derive a set of quasi-equilibrium equations that include a complete representation of the coriolis and metric terms. The end result may be thought of as a 'quasi-hydrostatic, quasi-equilibrium' model.

In order to generalize the results of the two previous subsections we introduce a 'geostrophic vertical velocity', which is denoted by w_g. Let $\mathbf{u}_g \leftrightarrow (u_g,\ v_g,\ w_g)$. Then following a similar calculation to that given in Dutton (1976), we have for the material derivative of the geostrophic flow

$$\frac{D\mathbf{u}_g}{Dt} = \mathbf{i}\frac{Du_g}{Dt} + \mathbf{j}\frac{Dv_g}{Dt} + \mathbf{k}\frac{Dw_g}{Dt} + u_g\frac{D\mathbf{i}}{Dt} + v_g\frac{D\mathbf{j}}{Dt} + w_g\frac{D\mathbf{k}}{Dt}\ , \tag{45}$$

where

$$\begin{aligned}
\frac{D\mathbf{i}}{Dt} &= u\left(\mathbf{j}\frac{\tan\phi}{r} - \frac{\mathbf{k}}{r}\right) \\
\frac{D\mathbf{j}}{Dt} &= -\mathbf{i}\frac{u\tan\phi}{r} - \frac{\mathbf{k}v}{r} \\
\frac{D\mathbf{k}}{Dt} &= \frac{\mathbf{i}u}{r} + \frac{\mathbf{j}v}{r}\ ,
\end{aligned}$$

which may be substituted back into (45) to obtain

$$\begin{aligned}
\frac{D\mathbf{u}_g}{Dt} &= \mathbf{i}\left(\frac{Du_g}{Dt} - \frac{uv_g\tan\phi}{r} + \frac{uw_g}{r}\right) \\
&+ \mathbf{j}\left(\frac{Dv_g}{Dt} + \frac{uu_g\tan\phi}{r} + \frac{vw_g}{r}\right) \\
&+ \mathbf{k}\left(\frac{Dw_g}{Dt} - \frac{uu_g}{r} - \frac{vv_g}{r}\right)\ .
\end{aligned} \tag{46}$$

We may now use (46) together with the $\cos\phi$ coriolis terms, in (38), to obtain

$$\left.\begin{aligned}\frac{Du_g}{Dt} - 2\Omega v\sin\phi - \frac{uv_g}{r}\tan\phi + 2\Omega w\cos\phi + \frac{uw_g}{r} + \frac{1}{r\rho\cos\phi}\frac{\partial p}{\partial\lambda} &= 0\\ \frac{Dv_g}{Dt} + 2\Omega u\sin\phi + \frac{uu_g}{r}\tan\phi + \frac{vw_g}{r} + \frac{1}{r\rho}\frac{\partial p}{\partial\phi} &= 0\\ -2\Omega u\cos\phi - \left(\frac{uu_g + vv_g}{r}\right) + g + \frac{1}{\rho}\frac{\partial p}{\partial r} &= 0\end{aligned}\right\} . \quad (47)$$

We remark that we may now set $w_g = 0$. In order to write the equations in η-coordinates (with η given by (15)), we use the chain rule as before and obtain

$$\begin{aligned}\frac{Du_g}{Dt} - 2\Omega v\sin\phi - \frac{uv_g}{r}\tan\phi + 2\Omega w\cos\phi + \frac{uw_g}{r}&\\ + \frac{1}{r\rho\cos\phi}\left(\frac{\partial p}{\partial\lambda} - \rho\mu'\frac{\partial r}{\partial\lambda}\right) &= 0 \end{aligned} \quad (48)$$

$$\frac{Dv_g}{Dt} + 2\Omega u\sin\phi + \frac{uu_g}{r}\tan\phi + \frac{vw_g}{r} + \frac{1}{r\rho}\left(\frac{\partial p}{\partial\phi} - \rho\mu'\frac{\partial r}{\partial\phi}\right) = 0 \quad (49)$$

$$\frac{\partial p}{\partial r} = \rho\mu' , \quad (50)$$

where

$$\mu' \equiv 2\Omega u\cos\phi + \left(\frac{uu_g + vv_g}{r}\right) - g \ . \quad (51)$$

The material derivative is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + \dot{\eta}\frac{\partial}{\partial\eta} \ , \quad (52)$$

and $\dot{\eta}$ may be diagnosed from (16). The continuity equation is the same as the non-hydrostatic model (21).

5 Hamiltonian Structure

5.1 *The non-hydrostatic equations*

We consider equations (12), (13) and (14) for adiabatic, frictionless flow. We follow Shepherd (1990) and write the thermodynamic equation in terms of the entropy S

$$S = C_p \log \theta = C_p \log \left[T \left(\frac{p}{p_0} \right)^{-\kappa} \right] \quad , \tag{53}$$

where p_0 is a constant reference pressure, C_p is the specific heat at constant pressure, and $\kappa \equiv \dfrac{R}{C_p}$. The system is Hamiltonian in the variables $\alpha^i = (u, v, w, \rho, S)$, with Hamiltonian functional

$$\mathcal{H} = \int \left[\frac{1}{2}\rho \left(u^2 + v^2 + w^2 \right) + E(\rho, S) + \rho g r \right] dr \; d\lambda \; d\phi \quad , \tag{54}$$

where E is the internal energy. If we write the absolute vorticity as

$$(\omega_1, \omega_2, \omega_3) \equiv \omega = \mathbf{\Omega} + \nabla \wedge \mathbf{u} \quad , \tag{55}$$

with $\mathbf{\Omega} = (0, 2\Omega \cos\phi, 2\Omega \sin\phi)$, then the equations of motion (12) and (13), with the conservation of entropy replacing (14), may be written in the Hamiltonian form

$$\frac{\partial \alpha^k}{\partial t} = \int d\mathbf{x} \, \frac{\delta \alpha^k}{\delta \alpha^i} J^{ij} \frac{\delta \mathcal{H}}{\delta \alpha^j} \quad , \tag{56}$$

where $d\mathbf{x}$ is a measure on configuration space. Summation on repeated indices is used and the bilinear, skew-symmetric matrix J^{ij} is given by

$$J^{ij} = \begin{pmatrix} 0 & \frac{1}{\rho}\omega_3 & -\frac{1}{\rho}\omega_2 & \frac{-1}{r\cos\phi}\frac{\partial}{\partial\lambda} & \frac{1}{r\rho\cos\phi}\frac{\partial S}{\partial\lambda} \\ -\frac{1}{\rho}\omega_3 & 0 & \frac{1}{\rho}\omega_1 & \frac{-1}{r}\frac{\partial}{\partial\phi} & \frac{1}{r\rho}\frac{\partial S}{\partial\phi} \\ \frac{1}{\rho}\omega_2 & -\frac{1}{\rho}\omega_1 & 0 & \frac{\partial}{\partial r} & \frac{1}{\rho}\frac{\partial S}{\partial r} \\ \frac{-1}{r\cos\phi}\frac{\partial}{\partial\lambda} & \frac{-1}{r}\frac{\partial}{\partial\phi} & -\frac{\partial}{\partial r} & 0 & 0 \\ \frac{-1}{\rho r\cos\phi}\frac{\partial S}{\partial\lambda} & \frac{-1}{\rho r}\frac{\partial S}{\partial\phi} & \frac{-1}{\rho}\frac{\partial S}{\partial r} & 0 & 0 \end{pmatrix} \quad . \tag{57}$$

It should be noted that the generic form of J^{ij} *is* skew despite the above, where, in fact, an integration by parts has been performed in (56) (this is not obvious from the result quoted in Shepherd (1990)). For an ideal gas $E = \rho C_v T$ (where $C_v = C_p - R$, is the specific heat at constant volume), whence

$$\delta E = C_v T\, \delta\rho + C_v \rho\, \delta T \ ,$$

and using

$$\delta T = \frac{T\, \delta S}{C_p - R} + \frac{RT\, \delta\rho}{(C_p - R)\rho} \ ,$$

we get

$$\delta E = C_v \rho \left(\frac{T}{C_v} \delta S + \frac{RT}{C_v \rho} \delta\rho \right) \ . \tag{58}$$

Using (58) we may verify the functional derivatives that are required in (56) are

$$\frac{\delta \mathcal{H}}{\delta \mathbf{u}} = \rho \mathbf{u}, \ \ \frac{\delta \mathcal{H}}{\delta S} = \frac{\partial E}{\partial S} = \rho T \tag{59}$$

$$\frac{\delta \mathcal{H}}{\delta \rho} = \frac{1}{2} |\mathbf{u}|^2 + gr + \frac{\partial E}{\partial \rho} = \frac{1}{2} |\mathbf{u}|^2 + gr + C_p T \ . \tag{60}$$

Using (53) we may verify that

$$-C_p \frac{\partial T}{\partial r} + T \frac{\partial S}{\partial r} = \frac{-1}{\rho} \frac{\partial p}{\partial r} \ ,$$

and thus demonstrate that (56) is precisely equivalent to the conventional form of the non-hydrostatic equations.

5.2 *The hydrostatic primitive equations*

Holm & Long (1989) formulate the ideal Boussinesq model with hydrostatic balance as a Hamiltonian system. They key idea is to use an isopycnal vertical coordinate, together with the continuity equation, to eliminate the explicit representation of vertical advection. In other words, under this transformation, the density becomes an independent variable whilst the height of an isopycnal surface becomes a dependent variable. Here, we invoke a similar idea, using isentropic coordinates in spherical geometry (Holm & Long worked with cartesian coordinates), and apply the method to the hydrostatic primitive equations. Like Holm & Long, we assume for our purposes here, that the potential temperature increases monotonically with height, although we do not rule out the possibility of the existence of a multi-region model with piecewise monotonicity.

We will start from the equations of motion written in isentropic coordinates

$$
\begin{aligned}
\frac{\partial u}{\partial t} &+ \frac{u}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\phi} \\
&- \left(2\Omega + \frac{u}{a\cos\phi}\right) v\sin\phi + \frac{1}{\rho a\cos\phi}\left(\frac{\partial p}{\partial\lambda} + g\rho\frac{\partial r}{\partial\lambda}\right) = 0 \qquad (61) \\
\frac{\partial v}{\partial t} &+ \frac{u}{a\cos\phi}\frac{\partial v}{\partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\phi} \\
&+ \left(2\Omega + \frac{u}{a\cos\phi}\right) u\sin\phi + \frac{1}{\rho a}\left(\frac{\partial p}{\partial\phi} + g\rho\frac{\partial r}{\partial\phi}\right) = 0 \qquad (62) \\
g &+ \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \ , \qquad (63)
\end{aligned}
$$

with the thermodynamic equation

$$
\frac{\partial\theta}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial\theta}{\partial\lambda} + \frac{v}{a}\frac{\partial\theta}{\partial\phi} = 0 \ , \qquad (64)
$$

and the continuity equation

$$
\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial\theta}\right) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}\left(u\frac{\partial p}{\partial\theta}\right) + \frac{1}{a}\frac{\partial}{\partial\phi}\left(v\frac{\partial p}{\partial\theta}\right) = 0 \ . \qquad (65)
$$

The isentropic potential vorticity is given by the expression

$$
q = -g\frac{\partial\theta}{\partial p}\left(2\Omega\sin\phi + \frac{1}{a\cos\phi}\left(\frac{\partial v}{\partial\lambda} - \frac{\partial(u\cos\phi)}{\partial\phi}\right)\right) \ . \qquad (66)
$$

We may identify a Hamiltonian functional

$$\mathcal{H} = \frac{-1}{g} \int \frac{\partial p}{\partial \theta} \left(\frac{1}{2} \left(u^2 + v^2 \right) + gr + C_p T \right) d\lambda \, d\phi \, d\theta \tag{67}$$

with functional derivatives taken with respect to the independent variables (u, v) and the inverse of the static stability $\dfrac{\partial p}{\partial \theta}$

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{-u}{g} \frac{\partial p}{\partial \theta}, \quad \frac{\delta \mathcal{H}}{\delta v} = -\frac{v}{g} \frac{\partial p}{\partial \theta} \tag{68}$$

$$\frac{\delta \mathcal{H}}{\delta \left(\frac{1}{g} \frac{\partial p}{\partial \theta} \right)} = - \left(\frac{1}{2} \left(u^2 + v^2 \right) + gr + C_p T \right) \ , \tag{69}$$

and write the equations of motion (61), (62)and (65) in the following explicit Hamiltonian form

$$\begin{pmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} \\ \dfrac{\partial}{\partial t} \left(\dfrac{1}{g} \dfrac{\partial p}{\partial \theta} \right) \end{pmatrix} = \begin{pmatrix} 0 & q & \dfrac{1}{a \cos \phi} \dfrac{\partial}{\partial \lambda} \\ -q & 0 & \dfrac{1}{a} \dfrac{\partial}{\partial \phi} \\ \dfrac{1}{a \cos \phi} \dfrac{\partial}{\partial \lambda} & \dfrac{1}{a} \dfrac{\partial}{\partial \phi} & 0 \end{pmatrix} \begin{pmatrix} \dfrac{-u}{g} \dfrac{\partial p}{\partial \theta} \\ \dfrac{-v}{g} \dfrac{\partial p}{\partial \theta} \\ - \left(\dfrac{1}{2} \left(u^2 + v^2 \right) + gr + C_p T \right) \end{pmatrix} \tag{70}$$

noting that we will need the result

$$C_p \frac{\partial T}{\partial c} = \frac{1}{\rho} \frac{\partial p}{\partial c}$$

which may be derived from

$$\nabla_\theta \log \theta = 0 \ ,$$

in order to obtain the form (61) and (62). The eulerian vertical velocity w may be diagnosed from

$$w = \frac{\partial r}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial r}{\partial \lambda} + \frac{v}{a} \frac{\partial r}{\partial \phi} \ . \tag{71}$$

Holm & Long (1989) have shown that their formulation shows an interesting analogy between the hydrostatic Boussinesq model and the equations for two dimensional, compressible barotropic flow. It is inappropriate to extend our discussion to such matters here, but we note that *if* the same analogies exist for the primitive equations then the (non-) linear stability theory for equilibrium solutions to the two-dimensional barotropic model, can, in principle, be carried over to the hydrostatic equations. We also note that the formulation (70) highlights the functional dependence of the skew-symmetric bilinear form (the symplectic structure) on the isentropic potential vorticity.

5.3 *The Quasi-Hydrostatic Equations*

In this subsection we give a heuristic account of what can be achieved using an isentropic formulation of the quasi-hydrostatic equations, together with the use of the zonal and meridional components of angular momentum on phase space. The ramifications of the functional methods will be presented in a more complete account elsewhere.

We begin by writing the equations (22) and (13) in isentropic coordinates. In the results that follow the local derivatives with respect to time and the angular variables are evaluated on θ surfaces. Note that extensive use is made of ($22)_3$ in the transformation that has taken place for the pressure gradient terms and for clarity we write everything out in full.

Equation (22)$_1$ becomes

$$\begin{aligned}
\frac{\partial u}{\partial t} &+ \frac{u}{r\cos\phi}\frac{\partial u}{\partial \lambda} + \frac{v}{r}\frac{\partial u}{\partial \phi} - 2\Omega v \sin\phi - \frac{uv}{r}\tan\phi \\
&+ 2\Omega\cos\phi\left(\frac{\partial r}{\partial t} + \frac{v}{r}\frac{\partial r}{\partial \phi}\right) + \frac{u}{r}\frac{\partial r}{\partial t} + \frac{uv}{r^2}\frac{\partial r}{\partial \phi} \\
&+ \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial \lambda} - \frac{v^2}{r^2\cos\phi}\frac{\partial r}{\partial \lambda} + \frac{g}{r\cos\phi}\frac{\partial r}{\partial \lambda} = 0 \ . \qquad (72)
\end{aligned}$$

Equation (22)$_2$ becomes

$$\begin{aligned}
\frac{\partial v}{\partial t} &+ \frac{u}{r\cos\phi}\frac{\partial v}{\partial \lambda} + \frac{v}{r}\frac{\partial v}{\partial \phi} + 2\Omega u \sin\phi - \frac{u^2}{r}\tan\phi \\
&+ \frac{v}{r}\left(\frac{\partial r}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial r}{\partial \lambda}\right) + \frac{1}{\rho r}\frac{\partial p}{\partial \phi} - \frac{1}{r}\left(2\Omega u\cos\phi + \frac{u^2}{r} - g\right)\frac{\partial r}{\partial \phi} = 0 \ , \qquad (73)
\end{aligned}$$

and the continuity equation becomes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial r}{\partial \theta}\right) + \frac{1}{r\cos\phi}\frac{\partial}{\partial \lambda}\left(\rho u\frac{\partial r}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\rho v\frac{\partial r}{\partial \theta}\right) = 0 \ . \qquad (74)$$

In the Hamiltonian formulation of (72)-(74) we need the zonal and meridional angular momentum coordinates; thus

$$\left.\begin{aligned}
m_1 &\equiv ru + \Omega r^2\cos\phi \\
m_2 &\equiv rv
\end{aligned}\right\} \ . \qquad (75)$$

We also require the isentropic potential vorticity which is given by

$$q = \left(-\frac{2\Omega\cos\phi}{r}\frac{\partial r}{\partial \phi} + 2\Omega\sin\phi + \frac{1}{r^2\cos\phi}\left(\frac{\partial(rv)}{\partial \lambda} - \frac{\partial(ru\cos\phi)}{\partial \phi}\right)\right)\frac{\partial \theta}{\partial r} \ . \qquad (76)$$

Consider an energy functional (Hamiltonian), written in terms of the angular momenta and isentropic coordinates:

$$\mathcal{H}\left[m_1, m_2, \rho\frac{\partial r}{\partial \theta}, T\right] = \int d\lambda d\phi d\theta \frac{\partial r}{\partial \theta}\rho\left\{\frac{1}{2}\left[\left(\frac{m_1}{r} - \Omega r\cos\phi\right)^2 + \left(\frac{m_2}{r}\right)^2\right] + C_pT + gr\right\} \ . \qquad (77)$$

This functional has the following variational derivatives,

$$\frac{\delta\mathcal{H}}{\delta m_1} = \rho\frac{u}{r}\frac{\partial r}{\partial \theta} \ , \quad \frac{\delta\mathcal{H}}{\delta m_2} = \rho\frac{v}{r}\frac{\partial r}{\partial \theta} \ , \qquad (78)$$

$$\frac{\delta \mathcal{H}}{\delta\left(\rho \frac{\partial r}{\partial \theta}\right)} = \frac{1}{2}\left(u^2 + v^2\right) + C_p T + g r \ . \tag{79}$$

We may now use (76), (78) and (79) to formulate the following result:

Proposition The equations of motion (72)-(74) may be written in the following Hamiltonian form

$$\begin{pmatrix} \frac{1}{r}\frac{\partial m_1}{\partial t} \\ \frac{1}{r}\frac{\partial m_2}{\partial t} \\ \frac{\partial}{\partial t}\left(\rho \frac{\partial r}{\partial \theta}\right) \end{pmatrix} = \begin{pmatrix} 0 & \frac{q}{\rho} & \frac{-1}{r\cos\phi}\frac{\partial}{\partial \lambda} \\ -\frac{q}{\rho} & 0 & \frac{-1}{r}\frac{\partial}{\partial \phi} \\ \frac{-1}{r\cos\phi}\frac{\partial}{\partial \lambda} & \frac{-1}{r}\frac{\partial}{\partial \phi} & 0 \end{pmatrix} \begin{pmatrix} r\frac{\delta \mathcal{H}}{\delta m_1} \\ r\frac{\delta \mathcal{H}}{\delta m_2} \\ \frac{\delta \mathcal{H}}{\delta\left(\rho \frac{\partial r}{\partial \theta}\right)} \end{pmatrix} \ . \tag{80}$$

Proof.

Explicit expansion of the matrix formulation and use variational derivatives given by (78) and (79). $\square$

This form of the equations, as with the primitive hydrostatic equations, emphasizes the role of the potential vorticity in the Hamiltonian structure. In order to accommodate the extra terms included in this approximation to a non-hydrostatic model, we have utilized a novel relationship between the angular momentum representation and the isentropic description. The important difference between these equations and the hydrostatic primitive equations from the point of view of a Hamiltonian formulation is that the local eulerian vertical velocity appears in the momentum equations. This means that blindly following the methods of Holm & Long as in §(5.2) leads nowhere. The use of angular momentum coordinates appears to be the key to solving this difficulty. As we mentioned at the beginning of this subsection, further details will be given elsewhere.

References

DUTTON, J. A. (1976) *The ceaseless wind; an introduction to the theory of atmospheric motion.* McGraw-Hill.

HOLM, D.D and LONG, B. (1989) Lyapunov stability of ideal stratified fluid equilibria in hydrostatic balance. *Nonlinearity*, **2**, 23-35.

JAMES, J.A. (1991) Transformation of the mesoscale model equations to a generalised vertical coordinate. *Short-range Forecasting Research Tech. Note* **57**.

KASAHARA, A. (1974) Various vertical coordinate systems used for numerical weather predication. *Mon. Wea. Rev.* **102**, 509-522.

PIELKE, R.A and MARTIN, C.L. (1981) The derivation of a terrain-following coordinate system for use in a hydrostatic model. *J. Atmos. Sci.* **38**, 1707-1713.

SHEPHERD, T.G. (1990) Symmetries, conservation laws and Hamiltonian structure in geophysical fluid dynamics. *Adv. Geophys.* **32**, 287-338.

WHITE, A.A and BROMLEY, R.A. (1990) Global quasi-hydrostatic models having a complete representation of the coriolis force. *Preprint.*

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